Curvature

Fix (M, E, ∇).

Recall de Rham complex: $0 \to C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \cdots$

$\omega \in \Omega^1(M)$: $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$

$d^2 = 0$

(Coordinate meaning: $\left.\begin{array}{l}(d\omega)_{ij} = \partial_i \omega_j - \partial_j \omega_i \\ (df)_i = \partial_i f\end{array}\right\} \to d^2 f = \partial_i \partial_j f - \partial_j \partial_i f = 0$)

Similarly have de Rham <u>coupled</u> to E: $0 \to \mathcal{E}(E) \xrightarrow{\nabla} \mathcal{E}(E \otimes T^*) \xrightarrow{d_\nabla} \mathcal{E}(E \otimes \Lambda^2 T^*) \to \cdots$

$\omega \in \mathcal{E}(E \otimes T^*)$: $d_\nabla \omega(X,Y) = \nabla_X \omega(Y) - \nabla_Y \omega(X) - \omega([X,Y])$

$d_\nabla \circ \nabla = F_\nabla : \mathcal{E}(E) \to \mathcal{E}(E \otimes \Lambda^2 T^*)$ <u>not</u> nec. zero anymore!

(More concretely: $F_\nabla(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} : \mathcal{E}(E) \to \mathcal{E}(E)$)

And $F_\nabla(fs) = f F_\nabla(s)$ for $f \in C^\infty(M)$, $s \in \mathcal{E}(E)$ (because $\nabla_X \nabla_Y (fs) - \nabla_Y \nabla_X (fs) - \nabla_{[X,Y]}(fs) = f\nabla_X \nabla_Y(s) - f\nabla_Y \nabla_X(s) - f\nabla_{[X,Y]} s$)

so F_∇ is a <u>section</u> of $\mathrm{Hom}(E, E \otimes \Lambda^2 T^*) = \mathrm{End}(E) \otimes \Lambda^2 T^*$

How to <u>compute</u> F_∇?

If $\nabla - \nabla' = A$ let's compute $F_\nabla - F_{\nabla'}$.

First note: $d_\nabla \omega = d_{\nabla'} \omega + A \wedge \omega$ (coords: $(d_\nabla \omega)^a = (d_{\nabla'} \omega)^a + A^a_b \wedge \omega^b$)

Thus
$$(d_\nabla \circ \nabla)s = d_{\nabla'}(\nabla s) + A \wedge (\nabla s)$$
$$= d_{\nabla'}(\nabla' s + As) + A \wedge (\nabla' s + As)$$
$$= d_{\nabla'}(\nabla' s) + d_{\nabla'}(As) + A \wedge \nabla' s + A \wedge (As)$$

<u>not</u> zero! because A is <u>End(E)-valued</u> 1-form

and $d_{\nabla'}$ obeys Leibniz rule: $d_{\nabla'}(As) = d_{\nabla'}(A)s - A \wedge d_{\nabla'} s$, so altogether

$$F_\nabla s = F_{\nabla'} s + (d_{\nabla'} A + A \wedge A)s$$

<u>In particular</u>: can always arrange $F_{\nabla'} = 0$ at least <u>locally</u>.

In that case get $F_\nabla s = (d_{\nabla'} A + A \wedge A) s$

ie $\quad F_\nabla = d_{\nabla'} A + A \wedge A \in \mathcal{E}(\operatorname{End} E \otimes \Lambda^2 T^*)$

This representation of F_∇ is local, but $F_\nabla \in \mathcal{E}(\operatorname{End} E \otimes \Lambda^2 T^*)$ is global.

If we pick a local basis where ∇' is the trivial conn, then A is the connection 1-form representing ∇ and we get $(F_\nabla)^a{}_b = dA^a_b + A^a_c \wedge A^c_b$.

$$\left(F_{\nabla\, ij}{}^a{}_b = \partial_i A_j{}^a{}_b - \partial_j A_i{}^a{}_b + A^a{}_{ci} A^c{}_{bj} - A^a{}_{cj} A^c{}_{bi} \right)$$

[<u>Basic example</u>: E of rank $r=1$. Then $F_\nabla = dA$ and a basis change by C leads to $A' = A + d \log C$, under which F_∇ is evidently invariant.]

Suppose $\nabla s = 0$. Then in particular $\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\, s = 0 \quad \forall X, Y$

i.e. $\quad s \in \operatorname{Ker} F(X,Y) \quad \forall X, Y$

Thus $F \neq 0$ is an <u>obstruction</u> to the existence of a basis $\{e_a\}$ with $\nabla e_a = 0$.

Converse is also true on $\mathbb{R}^n$:

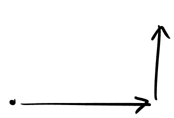
<u>Def</u> ∇ is <u>flat</u> if $F_\nabla = 0$.

<u>Prop</u> If $M = \mathbb{R}^n$ and (E, ∇) over M, with ∇ flat, <u>then</u> $\exists$ a basis $\{e_a\}$ of E with $\nabla e_a = 0$.

<u>Pf</u> Show it just for $n=2$.

Fix a basis $\{e_a(0,0)\}$ of $E|_{(0,0)}$. Then define $e_a(x^1, 0)$ by parallel xport from $(0,0)$.

Then define $e_a(x^1, x^2)$ by parallel xport from $(x^1, 0)$.



Need to show: $\quad \nabla_{\frac{\partial}{\partial x^1}} e_a = 0$

But $\nabla_{\frac{\partial}{\partial x^1}} \Big(\overset{0}{\cancel{\nabla_{\frac{\partial}{\partial x^2}} e_a}} \Big) - \nabla_{\frac{\partial}{\partial x^2}} \Big(\nabla_{\frac{\partial}{\partial x^1}} e_a \Big) = \overset{0}{\cancel{F\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right)}}\, e_a$

So $\nabla_{\frac{\partial}{\partial x^1}} e_a$ is parallel in the x^2-direction.

And it vanishes at $x^2 = 0$!

So $\nabla_{\frac{\partial}{\partial x^1}} e_a = 0$. ✓

If we choose this basis then ∇ would have connection coeff. $A = 0$.

In this sense, "a flat connection is locally trivial."

Another aspect of this:

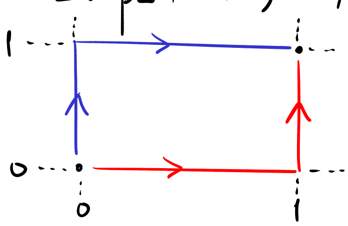
Thm If ∇ is flat then $P_{\gamma,\nabla}$ depends only on the homotopy class of γ.

Pf $\gamma(s,t) : [0,1] \times [0,1] \longrightarrow M$ $\qquad \gamma_0 = \gamma(0,\cdot) \qquad \gamma(s,0) = p_0$

$\gamma^*\nabla$ is flat over $[0,1]\times[0,1]$. $\qquad \gamma_1 = \gamma(1,\cdot) \qquad \gamma(s,1) = p_1$

Thus any $e_0 \in E_{p_0}$ can be extended to a $\gamma^*\nabla$-const. section $e(s,t)$ over $[0,1]\times[0,1]$.

In particular, $e(s,t)$ is still const. when restricted to either of these paths



so $P_{\gamma_0,\nabla}(e_0) = e(1,1)$

but also $P_{\gamma_1,\nabla}(e_0) = e(1,1)$ ∎

Cor If ∇ is flat and γ is contractible then $P_{\nabla,\gamma} = \mathbb{1}$.

On the other hand:

Prop If $r=1$ and $\gamma = \partial C$, then $P_{\nabla,\gamma} = e^{-\int_C F_\nabla}$.

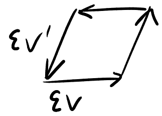
Pf $P_{\nabla,\gamma} = e^{-\oint_\gamma A} = e^{-\int_C dA} = e^{-\int_C F_\nabla}$. [Exercise: fill in details]

In $r>1$ case, it's not so easy to write $P_{\nabla,\gamma}$ in terms of F. Indeed, even when $F=0$, $P_{\nabla,\gamma}$ really depends on $\gamma \in \pi_1(M)$, not only $\gamma \in H_1(M)$.

But there is still an <u>infinitesimal</u> version:

<u>Prop</u> (Ambrose-Singer?)

Fix $v, v' \in T_pM$ and draw a parallelogram γ



$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\left[P_{\nabla,\gamma} - 1\right] = F_\nabla(v, v') \in \operatorname{End}(E_p)$$