Riemann normal coordinates

Prop M Riemannian:

$\forall p \in M$, $\exists$ nbhd U of $0 \in T_pM$, nbhd U' of $p \in M$ s.t. $\exp: U \to U'$ is diffeo.

Pf $\exp_*: T_0 T_pM \to T_pM$ is the identity map. (Because exp maps straight line $t \mapsto tv$ to a geodesic $t \mapsto \gamma_v(t)$ which has $\dot{\gamma}_v(0) = v$)

$T_0 T_pM$ is T_pM

In ptic $\exp_*: T_0T_pM \xrightarrow{\sim} T_pM$. So, apply inverse function thm $\Rightarrow$ exp is local diffeo.

Thus any basis $\{e_i\}$ for T_pM induces a local coord. sys on $U' \subset M$,

$$\exp(x^i e_i) \longmapsto (x^1, \dots, x^n)$$

If $\{e_i\}$ are orthonormal then call these Riemann normal coordinates.

In these coordinates, $g_{ij}(p) = \delta_{ij}$ $\quad \Gamma_{ij}^{k}(p) = 0$ $\leftarrow$ reflects non-tensorial nature of Γ!

$\partial_k g_{ij}(p) = 0$ [Exercise]

So we can find coords to make any Riem metric trivial to first order around a point.