<u>Def</u> W is a <u>uniformly normal nbhd</u> if $\exists \delta > 0$ s.t. $\forall q \in W$, W is contained in a geodesic δ-ball around q.

<u>Lemma</u> Given a nbhd U of p, $\exists$ a uniformly normal neighborhood W of p, with $W \subset U$.

<u>Pf</u> Consider $F: D \to M \times M$
$(q,v) \mapsto (q, \exp v)$.

$dF_{(p,0)}$ is invertible — indeed, in normal coords. on M, it would be represented by a matrix of the block form $\left(\begin{array}{c|c} 1 & * \\ \hline 0 & 1 \end{array}\right)$.

Thus, by inverse function thm, $\exists \underset{\substack{\cup \\ (p,0)}}{V} \subset D$ open, s.t. $F|_V$ is diffeo.

For $\mathcal{Y} \subset D$ <u>define</u> $\mathcal{Y}_\delta = \{(q,v) : q \in \mathcal{Y}, \|v\| < \delta\} \subset TM$.

① <u>Want</u> $\mathcal{Y}, \delta$ s.t. $\underset{\substack{\cup \\ (p,0)}}{\mathcal{Y}_\delta} \subset V$. Fix a coordinate patch $U' \ni p$ and let h be <u>Euclidean</u> metric in these coords. Since V is open it contains some set of the form $\{(q,v) : q \in \mathcal{Y}, \|v\|_h < \varepsilon\}$ with $\underset{\substack{\cup \\ p}}{\mathcal{Y}} \subset M$ open. But $\exists c > 0$ s.t. $\|v\|_h > c\|v\|$. So V contains $\mathcal{Y}_\delta$ with $\delta = \varepsilon / c$.
Shrinking $\mathcal{Y}$ if needed, can get $\mathcal{Y} \subset U$.

② $F(\underset{\substack{\cup \\ (p,p)}}{\mathcal{Y}_\delta})$ is open, so, $\exists$ $W \subset M$ open s.t. $\underset{\substack{\cup \\ (p,p)}}{W \times W} \subset F(\mathcal{Y}_\delta)$.

③ This W is the desired uniformly normal nbhd of p:
indeed, for any $q \in W$, have $\{q\} \times W \subset F(\mathcal{Y}_\delta)$, i.e. W is in the geodesic δ-ball around q.
Also any $q \in W$ has $(q,q) \in F(\mathcal{Y}_\delta)$, so $q \in \mathcal{Y}$; thus $W \subset \mathcal{Y} \subset U$ as desired.