# Jacobi fields

$\Gamma : (-\varepsilon, \varepsilon) \times (0, T) \longrightarrow M$ family of curves

$\quad\quad\;\; \rotatebox{90}{$\in$}\qquad\quad \rotatebox{90}{$\in$}$

$\quad\quad\;\; s \qquad\quad\;\; t$

**Thm** Suppose $\Gamma(s, \cdot)$ is a geodesic $\forall s$.
Let $\gamma = \Gamma(0, \cdot)$, and $V = \left.\frac{\partial \Gamma}{\partial s}\right|_{s=0}$ vector field along $\gamma$. ("variation field of a variation through geodesics")

Then $\nabla_t^2 V + R(V, \dot\gamma)\dot\gamma = 0$ (Jacobi equation).

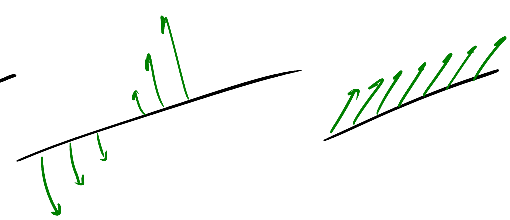
**Pf** Say $S = \Gamma_*\left(\frac{\partial}{\partial s}\right)$, $T = \Gamma_*\left(\frac{\partial}{\partial t}\right)$.

$$0 = \nabla_S \nabla_T (T) = \nabla_T \nabla_S (T) + R(S,T)T$$
$$= \nabla_T \nabla_T (S) + R(S,T)T$$

and evaluate at $s=0$, where $S = V$ and $T = \dot\gamma$.

**Def** If $\gamma$ is a geodesic and $\nabla_t^2 J + R(J, \dot\gamma)\dot\gamma = 0$ then say $V$ is a <u>Jacobi field along $\gamma$</u>.

**Ex** If $M = \mathbb{R}^n$, Jacobi fields have $\ddot J = 0 \Rightarrow$ <u>affine-linear</u>
$(V(t) = at + b)$



**Lemma** Every Jacobi field is the variation field of a variation through geodesics.

**Pf** Take $\Gamma(s,t) = \exp(t \cdot \sigma(s))$ where $\sigma : (-\varepsilon, \varepsilon) \longrightarrow TM$ has $\left[\begin{array}{l} \frac{d}{ds}(\pi \circ \sigma) = J(0) \\ \nabla_s \sigma = \nabla_t J(0) \end{array}\right]$

Then check $\partial_s \Gamma(s,t)\big|_{(0,0)} = J(0)$, $\nabla_t \partial_s \Gamma(s,t)\big|_{(0,0)} = \nabla_t J(0)$.

**Prop** $\gamma$ geodesic, $p = \gamma(t)$.
$\forall X, Y \in T_pM \quad \exists$ a Jacobi field $J$ with $J(p) = X$, $\nabla_t J(p) = Y$.

**Pf** Fix coordinates and use existence-uniqueness for $2^{nd}$-order ODE.

So the space of Jacobi fields on a given geodesic has dim $= 2n$.

Two are "trivial": 1) $J = \dot\gamma(t)$, 2) $J(t) = t\dot\gamma(t)$

They correspond to 1) $\Gamma(s,t) = \gamma(t+s)$, 2) $\Gamma(s,t) = \gamma(e^s t)$

$\Rightarrow$ just reparameterizations of $\gamma$.

**Def** Jacobi field $J$ is tangential if $J(t) \sim \dot\gamma(t)$ $\forall t$.

**Def/Prop** For Jacobi field $J$, TFAE: 1) $\forall t$ $\langle J(t), \dot\gamma(t)\rangle = 0$

2) $\exists t_1, t_2$ s.t. $\langle J(t_1), \dot\gamma(t_1)\rangle = \langle J(t_2), \dot\gamma(t_2)\rangle = 0$

3) $\exists t_0$ s.t. $\langle J(t_0), \dot\gamma(t_0) = 0\rangle$ and $\langle \nabla_t J(t_0), \dot\gamma(t_0)\rangle = 0$

In this can call $J(t)$ normal.

**Pf** Let $f(t) = \langle J(t), \dot\gamma(t)\rangle$.

$$\frac{d}{dt} f(t) = \langle \nabla_t J(t), \dot\gamma(t)\rangle$$

and
$$\begin{aligned}\frac{d^2}{dt^2} f(t) &= \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} J(t), \dot\gamma(t)\rangle \\ &= -\langle R(J,\dot\gamma)\dot\gamma, \dot\gamma\rangle \quad \text{by Jacobi eq.} \\ &= -Rm(J,\dot\gamma,\dot\gamma,\dot\gamma) \\ &= 0\end{aligned}$$

But if $\ddot f(t) = 0$ then $f(t) = 0\ \forall t \iff f(t_1) = f(t_2) = 0 \iff f(t_0) = \dot f(t_0) = 0$.

**Cor** $\{\text{Jacobi fields on } \gamma\} \simeq \underbrace{\{\text{tangential Jacobi fields}\}}_{\dim = 2} \oplus \underbrace{\{\text{normal Jacobi fields}\}}_{\dim = 2n-2}$

**Lemma** Fix $\gamma$ geodesic beginning at $p$, $W \in T_pM$

$J$ = Jacobi field on $\gamma$ with $J(p) = 0$, $\nabla_t J(p) = W$.

In normal coords around $p$, $J^i(t) = tW^i$

**Pf** Easy to see this $J$ obeys $J(p) = 0$ and $\nabla_t J(p) = W$ (use $\Gamma(p) = 0$ in normal coords)

Just need to check that this $J$ indeed obeys Jacobi eq.

$\gamma^i(t) = tV^i$ for some $V$. i.e. $\gamma(t) = \exp tV$

Define $\Gamma(s,t)$ by $\Gamma^i(s,t) = t(V^i + sW^i)$ i.e. $\Gamma(s,t) = \exp t(V+sW)$

That's a variation through geodesics, with $\left.\frac{\partial}{\partial s}\right|_{s=0} \Gamma(s,t) = tW^i$, so $tW^i$ is indeed Jacobi field.

**Lemma** Say $M$ has constant curvature $C$.

Then the normal Jacobi fields on $\gamma$ vanishing at $t=0$ are of the form

$$J(t) = u(t)\,E(t)$$

where $\nabla_t E(t) = 0$ and
$$u(t) = \begin{cases} t & C=0 \\ R\sin(\tfrac{t}{R}) & C = 1/R^2 \\ R\sinh(t/R) & C = -1/R^2 \end{cases}$$

**Pf** $R(X,Y)Z = C(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$ [just check this indeed has $K=C$]

so normal Jacobi fields have
$$\begin{aligned} 0 &= \nabla_t^2 J + R(J,\dot\gamma)\dot\gamma \\ &= \nabla_t^2 J + CJ \end{aligned}$$

Then try $J(t) = u(t)E(t)$ with $\nabla_t E(t) = 0$, $E(t) \perp \dot\gamma$

This works if $\ddot u(t) = C u(t)$ and $u(0)=0$; then $u$ is as above.

Varying the choice of $E(t)$ gives an $(n-1)$-dimensional space of Jacobi fields vanishing at $t=0$, which is all of them.

Combining these:

**Prop** Say $M$ has constant curvature $C$.

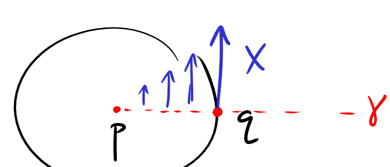
Fix normal coords around $p$, let $h$ be Euclidean metric in these coords.

Decompose vectors $V = V^T + V^\perp$ (tangent, normal to geodesic spheres)

Then
$$\|V\|_g^2 = \begin{cases} \|V^\perp\|_h^2 + \|V^T\|_h^2 & C=0 \\ \|V^\perp\|_h^2 + \frac{R^2}{r^2}\left(\sin^2\frac{r}{R}\right)\|V^T\|_h^2 & C = 1/R^2 \\ \|V^\perp\|_h^2 + \frac{R^2}{r^2}\left(\sinh^2\frac{r}{R}\right)\|V^T\|_h^2 & C = -1/R^2 \end{cases}$$

**Pf** Know $\|\frac{\partial}{\partial r}\|_g = \|\frac{\partial}{\partial r}\|_h = 1$. So, $\|V^\perp\|_g = \|V^\perp\|_h$.

$\Rightarrow$ Just need to compute $\|V^T\|_g^2$.




Say $X \in T_qM$, $X = X^T$; $\gamma$ geodesic from $p$ to $q$; let $r = r(q)$.

Then $X$ is value of a normal Jacobi field on $\gamma$, $J^i(t) = \frac{t}{r} X^i = u(t) E^i(t)$, $u(t)$ as above, $\nabla_t E = 0$.

Thus $\|X\|_g^2 = |u(r)|^2 \|E(r)\|_g^2 = |u(r)|^2 \|E(0)\|_g^2 = |u(r)|^2 \|E(0)\|_h^2$

and $\partial_t \left(\frac{t}{r} X^i\right) = \partial_t \left(u(t) E^i(t)\right) \Longrightarrow E^i(0) = \frac{1}{r} X^i$ (using $\dot{u}(0) = 1$) so $\|E(0)\|_h^2 = \frac{1}{r^2} \|X\|_h^2$

so altogether $\|X\|_g^2 = \frac{|u(r)|^2}{r^2} \|X\|_h^2$ ✓

Cor Say $M, M'$ have constant sec. curv. $C$.

Then $\forall p \in M, p' \in M'$, $\exists$ a nbhd of $p$ isometric to a nbhd of $p'$.

Pf Just take normal coords around both points.