Thm $\gamma : [0,T] \to M$ geodesic
(fixed endpoints)
$\Gamma : (-\varepsilon,\varepsilon)\times[0,T] \to M$ proper variation thru piecewise smooth curves
$V = \partial_s \Gamma|_{s=0}$ variation field

$$\frac{d^2}{ds^2} L(\Gamma_s) = \int_0^T \|\nabla_t V^\perp\|^2 - Rm(V^\perp, \dot\gamma, \dot\gamma, V^\perp)\, dt$$

Pf For simplicity, suppose Γ variation thru smooth curves.

Already showed $\displaystyle \frac{d}{ds} L(\Gamma_s) = \int_0^T \frac{\langle \nabla_t S, T\rangle}{\|T\|}\, dt$

Thus
$$\frac{d^2}{ds^2} L(\Gamma_s) = \int \frac{\langle \nabla_s \nabla_t S, T\rangle}{\|T\|} + \frac{\langle \nabla_t S, \nabla_s T\rangle}{\|T\|} - \frac{\langle \nabla_t S, T\rangle\langle \nabla_s T, T\rangle}{\|T\|^3}\, dt$$
$$= \int \frac{\langle \nabla_t \nabla_s S + R(S,T)S, T\rangle}{\|T\|} + \frac{\langle \nabla_t S, \nabla_t S\rangle}{\|T\|} - \frac{\langle \nabla_t S, T\rangle^2}{\|T\|^3}\, dt$$

[Take $s=0$: then $\|T\|=1$, $S=V$, $T=\dot\gamma$]
$$= \int \langle \nabla_t \nabla_s V, \dot\gamma\rangle - Rm(V,\dot\gamma,\dot\gamma,V) + \|\nabla_t V\|^2 - \langle \nabla_t V, \dot\gamma\rangle^2$$

Now the first term integrates to $\langle \nabla_s V, T\rangle \Big|_{t=0}^{t=T} = 0$ since variation is <u>proper</u>

and rest is $= \int -Rm(V^\perp, \dot\gamma, \dot\gamma, V^\perp) + \|\nabla_t V^\perp\|^2$ after decomposing $V = V^{\parallel} + V^\perp$

Similar for piecewise-smooth variations.

Def V, W proper normal vector fields
$$I(V,W) = \int_0^T \langle \nabla_t V, \nabla_t W\rangle - Rm(V,\dot\gamma,\dot\gamma,W)\, dt$$

Cor Γ proper variation w/ variation field $V \Rightarrow \displaystyle \frac{d^2}{ds^2}\Big|_{s=0} L(\Gamma_s) = I(V,V)$

I is like "Hessian" of the length.

Prop γ geodesic

V, W piecewise-smooth proper normal vector fields along γ, fail to be smooth at $t = a_1, \ldots, a_k$

δ_i = jump of $\nabla_t V$ at $t = a_i$

Then $$I(V,W) = -\int_0^T \langle \nabla_t^2 V + R(V,\dot\gamma)\dot\gamma, W\rangle - \sum_{i=1}^k \langle \delta_i, W(a_i)\rangle$$

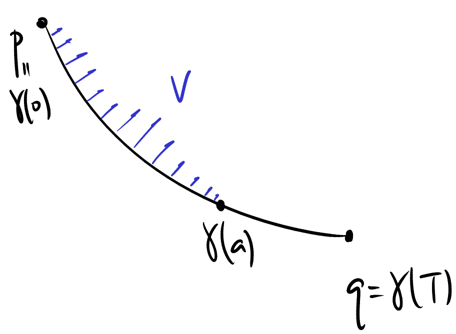
Pf Integration by parts:

$$\int_{a_{i-1}}^{a_i} \langle \nabla_t V, \nabla_t W\rangle \, dt = -\int_{a_{i-1}}^{a_i} \langle \nabla_t^2 V, W\rangle + \langle \nabla_t V, W\rangle \Big]_{a_{i-1}}^{a_i}$$

Thm γ geodesic from p to q with interior conjugate point to p

$\Longrightarrow$ γ is not minimizing.

Pf Say the conjugate point is $\gamma(a)$. Let J be a Jacobi field on $\gamma|_{[0,a]}$. Fix:



$$V(t) = \begin{cases} J(t) & 0 \le t \le a \\ 0 & t > a \end{cases}$$

Let δ be jump of $\nabla_t V$ at $t = a$. $(J \neq 0 \Rightarrow \delta \neq 0)$

Let W be any v.f. on γ s.t. $W(a) = \delta$.

Then take $X_\varepsilon = V + \varepsilon W$ ("rounding the corner")

$$I(X_\varepsilon, X_\varepsilon) = \underset{0}{I(V,V)} + 2\varepsilon \underset{<0}{I(V,W)} + \varepsilon^2 I(W,W)$$

so $I(X_\varepsilon, X_\varepsilon) < 0$ for ε small enough!

Rk More is known: # negative directions (Morse index) = # interior conjugate points!