Comments on the scalar Laplacian

In particular, we have formally self-adjoint operator Δ on $L^2(M)$.

Rk $\Delta f = -\operatorname{div} \operatorname{grad} f$ [Exercise]

In finite dimensions, we'd expect Δ admits ON-basis of eigenvectors.
Here too, it's true:

Thm If M compact, $L^2(M)$ has a countable ON-basis $\{f_n\}$ with $\Delta f_n = \lambda_n f_n$
and $\{\lambda_n\}$ has no accumulation points.

[Ex If $M = S^1$ this is the theory of Fourier series: $f_n = e^{inx/R}$, $\lambda_n = \frac{n^2}{R^2}$
If $M = S^2$ " " " " " spherical harmonics: eigenvalue $\frac{\ell(\ell+1)}{R^2}$ occurs with multiplicity $2\ell+1$]

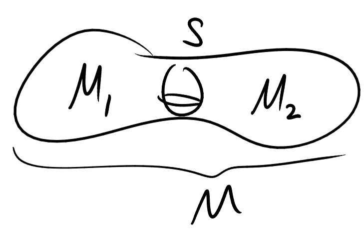
Pf Analysis! See Jost, sorry.

Some cool facts about Δ:

Thm [Weyl] Let $N(\lambda) = \#$ eigenvalues $\leq \lambda$ (w/ multiplicity)
As $\lambda \to \infty$, $N(\lambda) \sim \left[\frac{\mathrm{vol}(\text{unit ball in } \mathbb{R}^n)}{(2\pi)^n}\right] \mathrm{vol}(M) \cdot \lambda^{n/2}$

(So, you can "hear" the dimension and volume of M. But, you can't "hear" all of the metric: $\exists\, M_1, M_2$
which are not isometric but have same λ_n)

Thm [Cheeger] Let $h(M) = \inf_S \frac{\mathrm{vol}(S)}{\min[\mathrm{vol}(M_1), \mathrm{vol}(M_2)]}$



Then $\lambda_1 \geq \frac{1}{4} h(M)^2$

Thm [Lichnerowicz] If M compact, $\rho > 0$ s.t. $\mathrm{Ric}(X,X) \geq \rho \|X\|^2$ $\forall X \in TM$
then $\lambda_1 \geq \frac{n}{n-1}\rho$

<u>Rk</u> How to remember the <u>formula</u> for Δ: $\int \|df\|^2 \, vol = \int f \cdot \Delta f \, vol$

$$\int g^{ij} \partial_i f \partial_j f \sqrt{g} \, dx = - \int f \, \partial_i \left(\sqrt{g} \, g^{ij} \partial_j f \right) / \sqrt{g} \; \sqrt{g} \, dx$$

so $$\Delta f = - \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} \, g^{ij} \partial_j f \right)$$