Sobolev spaces

M Riemannian
E orthogonal v.b. over M, with orthogonal connection ∇

Def For $s \in \mathcal{E}_c(E)$, $\|s\|_{W_\ell} = \left[\int_M \|s\|^2 + \|\nabla s\|^2 + \cdots + \|\nabla^\ell s\|^2\right]^{1/2}$

compact supp.

Def $W_\ell(E) =$ completion of $\mathcal{E}_c(E)$ in the norm $\|\cdot\|_{W_\ell}$. (Banach space)

Bounded embeddings $W_0(E) \supset W_1(E) \supset W_2(E) \supset \cdots \supset C_c^\infty(E)$

$W_0(E) = L^2(E)$

Rk M compact $\Rightarrow$ 1) $W_\ell(E)$ does not depend on the metric in M or E, nor on ∇.
(changing these choices gives equivalent norms)
2) choosing a partition of 1, $1 = \sum \rho_i$, $s \mapsto \|s\|_{W_\ell}$ is equiv to $s \mapsto \sum_i \|\rho_i s\|_{W_\ell}$.

Rk Say $M = S^1$ w/ length 2π and E trivial line bundle; then $f = \sum_{p\in\mathbb{Z}} \hat{f}_p e^{ipx}$
has $\|f\|_{W_\ell}^2 = 2\pi \sum_p (1+p^2+\cdots+p^{2\ell}) |\hat{f}_p|^2$
so $f \in W_\ell(E) \Longleftrightarrow$ this sum converges $\Longleftrightarrow \sum_{p\in\mathbb{Z}} p^{2\ell}|\hat{f}_p|^2$ converges.

Using Fourier analysis, can define more generally $W_s(E)$ for any $s \in \mathbb{R}$ (and any M, E).

Prop A k^{th}-order differential operator $D: \mathcal{E}(E) \to \mathcal{E}(F)$
extends to a bounded $D: W_\ell(E) \to W_{\ell-k}(F)$, $\forall \ell \geq k$.

Pf Exercise.

Lemma (Rellich) $W_1(E) \hookrightarrow W_0(E)$ is compact.

Pf Want to show image of bounded sets is precompact, i.e. any bdd sequence in $W_1(E)$
has a subsequence which converges in $W_0(E)$.

Using partitions of unity and trivializ^{n}s, reduce to the same question for compactly supported functions on $\mathbb{R}^n$. This is $\simeq$ to the same question for compactly supported f^{n}s on $T^n = (S^1)^n$.

$$f(x) = \sum_p{}' \hat{f}_p e^{ipx} \qquad p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$$

$$\|f\|^2_{W_1} = \sum_p (1 + \|p\|^2) |\hat{f}_p|^2$$

Fix $\varepsilon > 0$.

Let $Z_N = \{f \in W_1(T^n) : \hat{f}_p = 0 \text{ for } \|p\| \leq N\}$ "high frequency part"

$(f \in Z_N \text{ and } \|f\|_{W_1} < 1) \Rightarrow \|f\|_{W_0} < \frac{1}{1+N^2}$. Fix N s.t. $\frac{1}{1+N^2} < \frac{\varepsilon}{\sqrt{2}}$.

Meanwhile $Z_N^\perp = \{f \in W_1(T^n) : \hat{f}_p = 0 \text{ for } \|p\| > N\}$ is finite-dim, so $\{f : f \in Z_N^\perp \text{ and } \|f\|_{W_1} < 1\}$ can be covered by finitely many $\|\cdot\|_{W_0}$-balls of radius $< \frac{\varepsilon}{\sqrt{2}}$.

Thus $\{\|f\|_{W_1} < 1\}$ can be covered by finitely many $\|\cdot\|_{W_0}$-balls of radius $< \varepsilon$.

So it's <u>totally bounded</u> subset of a <u>complete</u> metric space $\Rightarrow$ <u>precompact</u> (Heine-Borel) ✓

<u>Lemma</u> (Sobolev) If $l - \frac{n}{2} > k$, then $W_l(E) \hookrightarrow C^k(E)$.

<u>Pf</u> For $k = 0$:

Use partitions of unity to reduce to f^{n}s supported inside unit ball $B \subset \mathbb{R}^n$.

Suppose f <u>smooth</u>, then $f(0) = C \int_0^1 r^{l-1} \frac{\partial^l f}{\partial r^l}$ [$\int$ by parts in a fixed radial dir]

ie $\quad f(0) = C \int_B \frac{\partial^l f}{\partial r^l} r^{l-n} \cdot \mathrm{vol}$ [average over all radial dir]

$$|f(0)| \leq C \sqrt{\int_B \left|\frac{\partial^\ell f}{\partial r^\ell}\right|^2 \mathrm{vol}} \sqrt{\int_B r^{2(\ell-n)} \mathrm{vol}} \qquad [\text{Cauchy-Sch}]$$

$$\leq C' \|f\|_{W_\ell} \qquad [\ell > n/2]$$

So $\|f\|_{L^\infty} \leq C' \|f\|_{W_\ell}$

and $C^0(E)$ is the completion of $\mathcal{E}(E)$ in the L^∞ norm ✓
(uniform limit of continuous functions is continuous)

Similar for $k > 0$, just "differentiate under the $\int$ sign" in the above.

We also need:

Def $s \in \mathcal{E}(E)$: $\|s\|_{W_{-1}} = \min\{C : \langle s, s' \rangle_{L^2} \leq C \|s'\|_{W_1} \quad \forall s' \in W_1(E)\}$

Def $W_{-1}(E) =$ completion of $\mathcal{E}(E)$ wrt $\|\cdot\|_{W_{-1}}$

Rk elements of $W_{-1}(E)$ may be distributions which are not functions.

e.g. if $M = S^1$, E triv, $\delta(x) \in W_{-1}(E)$ [since we showed already that $\|f\|_{L^\infty} \leq C \|f\|_{W_1}$]

Prop $W_{-1}(E) \simeq W_1(E)^*$ [= space of bounded linear operators on $W_1(E)$]

Pf Cauchy sequence $\{s_n\}$ in $\mathcal{E}(E)$, $s_n \to s$ in $W_{-1}(E)$, $s' \in W_1(E)$:
then $\langle s_n, s' \rangle_{L^2}$ is also Cauchy sequence, call its limit $\langle s, s' \rangle$.
This gives $W_{-1}(E) \longrightarrow W_1(E)^*$. To see it's $\simeq$, need to know that this pairing is nondegenerate: just use the fact that it's nondegenerate on $\mathcal{E}(E)$ and $\mathcal{E}(E)$ is dense in both $W_1(E)$ and $W_{-1}(E)$.

Prop $W_0(E) \hookrightarrow W_{-1}(E)$ is bounded.

Pf $\langle s, s' \rangle_{L^2} \leq \|s\|_{L^2} \|s'\|_{L^2} \leq \|s\|_{L^2} \|s'\|_{W_1} \implies \|s\|_{L^2} \geq \|s\|_{W_{-1}}$.

Prop Any 2^{nd}-order diff. op. extends to a bounded map $W_1(E) \longrightarrow W_{-1}(E)$.

Pf ...