Prop $(\Delta+\kappa): W_1 \to W_{-1}$ is an isomorphism.

Pf $\langle (\Delta+\kappa)s, s\rangle_{L^2} \geq \|s\|^2_{W_1} \Rightarrow \Delta+\kappa$ is injective.

For surjectivity, need to show: any bounded linear functional on W_1 is of the form

$$s \mapsto \langle (\Delta+\kappa)s', s\rangle_{L^2}$$

To prove this:

The norm $\|s\|^2_{\tilde{W}_1} = \langle (\Delta+\kappa)s, s\rangle_{L^2}$ is equivalent to the usual one on W_1

($\|s\|_{\tilde{W}_1} \leq c\|s\|_{W_1}$ follows from boundedness of $\Delta: W_1 \to W_{-1}$, other direction is Gårding)

Also this norm comes from a Hilbert space structure: $\langle s, s'\rangle_{\tilde{W}_1} = \langle (\Delta+\kappa)s, s'\rangle_{L^2}$

Then can use Riesz Representation Thm.

Now consider $\underset{\text{bdd}}{W_0 \hookrightarrow W_{-1}} \underset{\text{bdd}}{\xrightarrow{(\Delta+\kappa)^{-1}}} \underset{\text{compact}}{W_1 \hookrightarrow W_0}$ This is a compact, self-adjoint operator on the Hilbert space W_0.

Thm Any compact self-adjoint operator T has a basis of eigenvectors

$$T v_n = \lambda_n v_n \qquad n = 1, 2, \ldots$$

with $\lim_{n\to\infty} \lambda_n = 0$.

Pf $v \mapsto \langle v, Tv\rangle$ attains maximum on the unit sphere at some v_1 (using compactness of T)

Then T preserves $v_1^\perp$ and $T|_{v_1^\perp}$ obeys all the hypotheses.

Proceed using AC to find maximal subspace spanned by eigenvectors...

So, we can diagonalize $(\Delta+\kappa)^{-1}$ on W_0.

But from there it's easy to diagonalize Δ on W_0, too [the basis of eigenvectors is the same!].

Next, want to see that the eigenvectors are actually C^∞. ("elliptic regularity" — recall that e.g. all harmonic functions are C^∞)

Let $E = \Lambda^*(T^*M)$ and $D = d + d^* : E \to E$. Note $\Delta = D^2$.

Prop ("elliptic estimate")

$\forall \ell$, $\exists C$ s.t. $\forall s \in W_{\ell+1}$, $\quad \|s\|_{W_{\ell+1}} \leq C\left(\|Ds\|_{W_\ell} + \|s\|_{W_\ell}\right)$

Pf Induction on ℓ

For $\ell = 0$, it follows from Gårding ineq:

$$\|s\|^2_{W_1} \leq \langle(\Delta + \kappa)s, s\rangle_{L^2} = \|Ds\|^2_{L^2} + \kappa\|s\|^2_{L^2}$$

$$\Longrightarrow \|s\|_{W_1} \leq \sqrt{\|Ds\|^2_{L^2} + \kappa\|s\|^2_{L^2}} \leq \|Ds\|_{L^2} + \sqrt{\kappa}\,\|s\|_{L^2}$$

For $\ell > 0$,

$$\|\nabla_i s\|_{W_\ell} \leq C\left(\|D\nabla_i s\|_{W_{\ell-1}} + \|\nabla_i s\|_{W_{\ell-1}}\right)$$

$$= C\left(\|\nabla_i D s\|_{W_{\ell-1}} + \|\underbrace{[D,\nabla_i]}_{\text{order zero operator}} s\|_{W_{\ell-1}} + \|\nabla_i s\|_{W_{\ell-1}}\right)$$

$$\leq C\left(\|Ds\|_{W_\ell} + \|s\|_{W_\ell}\right)$$

[compute in normal coords using $d = a_i^* \nabla_i$, $d^* = a_i \nabla_i$]

Cor $\forall \ell$, $\exists C$ s.t. $\forall s \in W_{\ell+2}$, $\quad \|s\|_{W_{\ell+2}} \leq C\left(\|\Delta s\|_{W_\ell} + \|s\|_{W_\ell}\right)$

Pf

$$\|s\|_{W_{\ell+2}} \leq C\left(\|Ds\|_{W_{\ell+1}} + \|s\|_{W_{\ell+1}}\right) \qquad \text{by elliptic estimate}$$

$$\leq C\left(\|D^2 s\|_{W_\ell} + \|Ds\|_{W_\ell} + \|s\|_{W_\ell}\right) \qquad \text{by elliptic estimate}$$

$$\leq C\left(\|D^2 s\|_{W_\ell} + \|D^2 s\|_{W_{\ell-1}} + \|Ds\|_{W_{\ell-1}} + \|s\|_{W_\ell}\right) \qquad \text{by elliptic estimate}$$

$$\leq C\left(\|D^2 s\|_{W_\ell} + \|s\|_{W_\ell}\right) \qquad \text{since } \|\cdot\|_{W_\ell} > \|\cdot\|_{W_{\ell-1}} \text{ and } D \text{ bdd}$$

Lemma Suppose $s \in W_\ell$ and $\Delta s \in W_\ell \subset W_{\ell-2}$. Then $\exists\ s_n \in \mathcal{E}$ such that $s_n \to s$ and $\Delta s_n \to \Delta s$ in W_ℓ.

Pf Use partition of unity to reduce to sections of trivial bundle, acted on by a 2^{nd}-order diff op Δ.

Use "mollifiers": let $\varphi \in C_c^\infty(\mathbb{R}^d)$, with $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$, supported on $\|x\|<1$, $\varphi_\varepsilon = \varepsilon^d \varphi\left(\frac{x}{\varepsilon}\right)$

$$s_n = s * \varphi_{\frac{1}{n}} = \int dy\ s(x-y)\varphi_{\frac{1}{n}}(y)$$

Then $\|s_n - s\|_{L^2} = \int dy \left[s(x-y) - s(x)\right] \varphi_{\frac{1}{n}}(y) \to 0$ (approximate s by C^∞ sections and use fact that $\varphi_{\frac{1}{n}}(y)$ supp. in $\|y\| < \frac{1}{n}$)

Similarly for the Sobolev norms.

Lemma Suppose $s \in W_\ell$ and $\Delta s \in W_\ell \subset W_{\ell-2}$. Then $s \in W_{\ell+2}$.

Pf Introduce s_n as in previous lemma.

$$\|s_n - s_m\|_{W_{\ell+2}} \leq C\left(\|s_n - s_m\|_{W_\ell} + \|\Delta s_n - \Delta s_m\|_{W_\ell}\right)$$

$\Longrightarrow (s_n)$ Cauchy in $W_{\ell+2}$, hence converges, i.e. $s \in W_{\ell+2}$.

Lemma Suppose $s \in W_\ell$ and $\Delta s \in \mathcal{E}$. Then $s \in \mathcal{E}$.

Pf By induction using the previous lemma, $s \in W_{\ell'}$ for all ℓ'.

By Sobolev lemma it follows that s is C^∞.

Finally we can prove the key missing step in the pf of Hodge thm:

Prop $\exists G : \Omega^p(M) \to \Omega^p(M)$ s.t. for $\omega \in (\operatorname{Ker}\Delta)^\perp$, $\Delta(G\omega) = \omega$.

Pf ω is L^2, so can expand $\omega = \sum c_n \omega_n$ in L^2, with $\Delta\omega_n = \lambda_n \omega_n$; and $\omega \in (\operatorname{Ker}\Delta)^\perp$ says all $\lambda_n \neq 0$. Then define $G\omega = \sum \frac{c_n}{\lambda_n}\omega_n$. This is only L^2 a priori, but also has $\Delta(G\omega) = \omega$, so in fact $G\omega$ is C^∞.