Elliptic Operators

In pf of Hodge thm we invert Δ away from its kernel, i.e. produce a "Green's op" $G : \Omega^p \to \Omega^p$ such that for $\varphi \perp \operatorname{Ker}\Delta$ we have $\varphi = \Delta(G\varphi)$.

This can be done rather generally for self-adjoint elliptic operators acting over compact manifolds.

To motivate this def, let's consider how we might try to produce G — e.g., for Δ = scalar Laplacian on $T^n = \mathbb{R}^n/\mathbb{Z}^n$.

$$f(x) = \sum_{p \in \mathbb{Z}^n} e^{-ip\cdot x}\tilde{f}(p).$$

$$f \text{ is } C^\infty \iff \sum_{\vec{p}} \|p\|^k |\tilde{f}(p)| < \infty \ \forall k.$$

$$Gf(\vec{x}) = \sum_{\vec{p} \in \mathbb{Z}^n} e^{-i\vec{p}\cdot\vec{x}} \frac{\tilde{f}(\vec{p})}{\|\vec{p}\|^2}.$$

$Gf(\vec{x})$ will make sense so long as $\tilde{f}(\vec{p}) = 0$.

But $\tilde{f}(\vec{p}) = \int_{T^n} d\vec{x}\, f(\vec{x}) = \langle 1, f\rangle$

i.e. $\tilde{f}(\vec{p}) = 0$ says that f is $\perp$ to 1, i.e. $\perp$ to $\mathcal{H}^0$

$f\ C^\infty \Rightarrow Gf\ C^\infty$.

This depended crucially on the fact that $\|p\|^2$ is positive definite!

Consider in contrast $\Delta' = \frac{\partial^2}{\partial x^2} - \alpha^2\frac{\partial^2}{\partial y^2}$.

- If $\alpha \in \mathbb{Q}$ then Δ' has infinite-dimensional kernel $\left(\alpha = \frac{p}{q} \text{ then } \Delta'(e^{ipx+iqy}) = 0\right)$

- If $\alpha \notin \mathbb{Q}$ but has good rational approx, i.e. $|p_n - q_n\alpha| < e^{-n}$,

  $f\ C^\infty \not\Rightarrow Gf\ C^\infty$ $\quad \left(f = \sum' \frac{e^{-n}}{n^2} e^{ip_nx+iq_ny} \text{ then } Gf = \sum' \frac{1}{p_n^2 - \alpha^2 q_n^2} \frac{e^{-n}}{n^2} e^{ip_nx+iq_ny}\right)$

Def

E, F C^∞ v.b. over M.

A differential operator of order k mapping E to F is a map $D: C^\infty(E) \to C^\infty(F)$ such that in any coord. chart and local trivialization

$$Ds = \sum_{|I| \leq k} a_I(x)\left[\frac{\partial}{\partial x_I} s(x)\right] \quad \left(\frac{\partial}{\partial x_I} = \frac{\partial}{\partial x_{I_1}} \cdots \frac{\partial}{\partial x_{I_n}}\right) \quad a_I \in \mathrm{Hom}(E, F)$$

Ex d, d^*, Δ

Let $\mathrm{Diff}_k(E,F)$ be the set of such operators.

(NB: $\mathrm{Diff}_k(E,F) \subset \mathrm{Diff}_{k'}(E,F)$ $\forall k' \geq k$.)

Def Say $M = \mathbb{R}^n$, $E = \mathbb{R}^a \times M$, $F = \mathbb{R}^b \times M$; $\quad T'M = T^*M \setminus (M \times \{0\})$; $\pi: T'M \to M$

The k-symbol of $D = \sum_{|I| \leq k} a_I(x) \frac{\partial}{\partial x_I}$ is $\sigma_k(D): \pi^*(E) \to \pi^*(F)$

ie $\sigma_k(D)(p,x): E_x \to F_x$

$$\sigma_k(D)\left((p,x), e\right) = i^k \sum_{|I|=k} p^I [a_I(x)](e).$$

(Homogeneous of degree k along the fibers of $T'M$.)

Def/Prop $D \in \mathrm{Diff}_k(E,F)$:

The principal symbol $\sigma_k(D)$ is the unique map $\pi^*(E) \longrightarrow \pi^*(F)$ which in every local coord. sys. agrees with the k-symbol.

Pf For $(x,p) \in T'M$, $e \in E_x$,

take some $g \in C^\infty(M)$ w/ $dg(x) = p$, $g(x) = 0$

and some $s \in C^\infty(M,E)$ w/ $s(x) = e$.

Then define $\sigma_k(D)(e) = \left[D\left(\frac{i^k}{k!} g^k s\right)\right](x) \in F_x$.

Have to check independence of choices! Then taking g linear, recover our local-coordinate formulas above. $\blacksquare$

Def $D \in \mathrm{Diff}_k(E,F)$:

D is *elliptic* if $\sigma_k(x,p)$ is *invertible* $\forall (x,p) \in T^*M$.

Everything we proved about the Laplacian has a close analog for any self-adj. elliptic operator over a compact manifold.

Ex $\quad d : \Omega^\cdot(M) \to \Omega^\cdot(M)$ is $d \in \mathrm{Diff}_1(\Lambda^\cdot(T^*M), \Lambda^\cdot(T^*M))$

It has $[\sigma_1(d)](x,p) = ip\wedge \cdot$

$\left[\text{Because } d(g\alpha) = dg \wedge \alpha + \overset{0 \text{ at } x}{\cancel{g(d\alpha)}}\right]$

Prop $D \in \mathrm{Diff}_k(E,F)$, M Riemannian:

D has a formal adjoint $D^* \in \mathrm{Diff}_k(F,E)$

and $\sigma_k(D^*) = [\sigma_k(D)]^*$

Pf Integration by parts in local coordinates.

Prop $D \in \mathrm{Diff}_k(E,F)$, $D' \in \mathrm{Diff}_{k'}(F,G)$: $D' \circ D \in \mathrm{Diff}_{k+k'}(E,G)$

$\sigma_{k+k'}(D' \circ D) = \sigma_{k}(D') \cdot \sigma_k(D)$

Pf Local coordinates.

$$\left(\begin{array}{ll} \text{e.g. } D' = f'(x)\dfrac{d}{dx},\ D = f(x)\dfrac{d}{dx} & \text{— } D'D = f'(x)f(x)\dfrac{d^2}{dx^2} + (\text{lower order}) \\ \sigma_1(D') = i f'(x) p \quad \sigma_1(D) = i f(x) p & \quad \sigma_2(D'D) = -f'(x)f(x)p^2 \end{array}\right)$$

Prop $\Delta : \Omega^p(M) \longrightarrow \Omega^p(M)$ is an elliptic operator.

Pf
$$\begin{aligned} \sigma_2(\Delta) &= \sigma_1(d)\sigma_1(d^*) + \sigma_1(d^*)\sigma_1(d) \\ &= -\left[(p\wedge\cdot)(p\lrcorner\cdot) + (p\lrcorner\cdot)(p\wedge\cdot)\right] \\ &= -\|p\|^2 \cdot \mathbb{1} \end{aligned}$$