Principal bundles

Def A *Lie group* is a group G which is also a manifold, such that
$$(g,g') \mapsto gg' \quad \text{and} \quad g \mapsto g^{-1} \quad \text{are smooth.}$$

$$L_g : G \to G \qquad\qquad R_g : G \to G$$
$$g' \mapsto gg' \qquad\qquad g' \mapsto g'g$$

Def $\mathfrak{g} = \mathrm{Lie}(G) = \{X \in TG : (L_g)_* X = X\}$

Prop
- If $X, Y \in \mathfrak{g}$ then $[X,Y] \in \mathfrak{g}$.
- $\mathfrak{g} \simeq T_1 G$.
  $X \mapsto X(1)$

Def $\forall g \in G$, $\mathrm{Ad}_g : \begin{array}{l} G \to G \\ g' \mapsto g g' g^{-1} \end{array}$ induces $\mathrm{Ad}_g : \underset{\mathfrak{g}}{T_1 G} \to \underset{\mathfrak{g}}{T_1 G}$

Def M manifold, G Lie group: G *acts smoothly on* M (from the right) if G acts on M (from the right) and
$$G \times M \to M, \quad (g,m) \mapsto mg$$
is smooth.

Rk If G acts smoothly on M, we get a homomorphism $\mathfrak{g} \to TM$, $X \mapsto \sigma(X)$

$$\sigma(X)(m) = \frac{d}{dt}(m \cdot g(t))$$

X, $g(t)$, $\dot{g}(0) = X$

Def M mfd, G Lie gp: a *principal G-bundle over M* is (P, π) where

1) P is a manifold acted on by G
2) $\pi : P \to M$ has $\pi(xg) = \pi(x)$
3) $\forall p \in M$, $\exists$ nbhd $\underset{p}{U \ni} \subset M$ and a *diffeomorphism* $\varphi_U : \pi^{-1}(U) \to U \times G$
   with $\varphi(x) = (\pi(x), \nu(x))$
   $\varphi(xg) = \varphi(x) g$

Ex $P = M \times G$ is a principal G-bundle over M ("trivial G-bundle").

Ex $G = \mathbb{Z}/2\mathbb{Z}$, P = Möbius covering. This P has no global section.

Rk · The fibers of P are G-torsors, not canonically $\simeq G$.

· Choosing a section $s: M \to P$ determines a trivialization of P:
$$M\times G \xrightarrow{\sim} P \qquad (p,g) \longmapsto s(p)\cdot g$$

Ex Say $G = GL(n,\mathbb{R}) = \{(g^j_i)_{i,j=1,\dots,n} \in \mathbb{R}^{n^2} \mid \det(g^j_i) \neq 0\}$

For $x \in M$, let $P_x = \{\text{bases } \{e_1,\dots,e_n\} \text{ of } T_xM\} = \{(e_1,\dots,e_n) \in T_xM \mid e_1\wedge\dots\wedge e_n \neq 0\}$

G acts on P_x by $\{e_i\} \to \{e_i' = g^j_i e_j\}$ [NB: not $e_i' = ge_i$, which would make sense for $g \in \operatorname{End} TM$ but doesn't for $g \in GL(n,\mathbb{R})$]

Define $P = \coprod_{x\in M} P_x$. This has structure of principal G-bundle over M. [Exercise]

Def Suppose G acts on Y. Then define associated bundle:

$$Y_P = P\times_G Y = P\times Y/_{(x,y)\sim(xg,\, g^{-1}y)} \quad \forall g \in G$$

$\pi: P \to M$ induces
$$\pi: P\times_G Y \to M \qquad [(x,y)] \mapsto \pi(x)$$
making $P\times_G Y$ a fiber bundle over M.

Prop If P is trivial, ie $P = M\times G$, then $P\times_G Y = M\times Y$.
$$[((m,g),y)] \mapsto (m, gy)$$

Roughly, in $P\times_G Y$ we "replace the G fibers by Y fibers." No G-action anymore!

Prop If E is a representation of G, then $P\times_G E$ is a vector bundle over M.

Ex P a principal $\mathbb{Z}/2\mathbb{Z}$-bundle, E = nontriv 1-d rep of $\mathbb{Z}/2\mathbb{Z}$, then $P\times_G E$ is a real line bundle.

Ex P bundle of frames, E = fundamental rep of $GL(n,\mathbb{R})$ then $P\times_G E \simeq TM$
$$[((m,e_1,\dots,e_n),v)] \mapsto (m, \textstyle\sum v_i e_i)$$

Prop • If E_1, E_2 are G-reps then
$$(E_1)_P \oplus (E_2)_P \simeq (E_1 \oplus E_2)_P$$
$$(E_1)_P \otimes (E_2)_P \simeq (E_1 \otimes E_2)_P$$
$$(E_P)^* \simeq (E^*)_P$$

operations on vector bundles — operations on G-representations

Pf Exercise.

So the bundles $T^p_q = \underbrace{T \otimes \cdots \otimes T}_{p} \otimes \underbrace{T^* \otimes \cdots \otimes T^*}_{q}$ we have used in this course are all <u>associated bundles</u> for the bundle of frames P.

<u>Maurer-Cartan form</u>

G Lie gp, $\mathfrak{g} = \mathrm{Lie}\, G = \{\text{left invariant vector fields on } G\}$

$\theta_G \in \mathcal{E}(\Lambda^* TG \otimes \mathfrak{g})$ defined by: $[\theta_G(g)](X) = L_{g^{-1}*} X \in T_1 G \simeq \mathfrak{g}$

or equivalently: $\theta_G(X) = X \quad \forall X \in \mathfrak{g}$

Prop For $G \subset GL(n, \mathbb{R})$, $\theta_G = g^{-1} dg$.

[upon identifying $\mathfrak{g} = T_1 G$ and using

$g : G \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{R})$, tautological; $dg : T_g G \longrightarrow \mathrm{Mat}_{n \times n}(\mathbb{R})$, i.e. $dg \in \mathcal{E}(T^*G \otimes \mathrm{Mat}_{n\times n}(\mathbb{R}))$; $g^{-1} : \mathrm{Mat}_{n\times n}(\mathbb{R}) \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{R})$, $M \longmapsto g^{-1}M$]

Pf Exercise.

Prop $d\theta_G + \theta_G \wedge \theta_G = 0$. (Maurer-Cartan)

Pf
$$d\theta_G(\sigma(X), \sigma(Y)) = \sigma(X) Y - \sigma(Y) X - [X, Y] \qquad \forall X, Y \in \mathfrak{g}$$
$$= -[X, Y] \quad \text{since } X, Y \in \mathfrak{g} \text{ are } \underline{\text{constant}}$$
$$= -[\theta_G(\sigma(X)), \theta_G(\sigma(Y))]$$

Rk Fix any rep $\rho: G \to \mathrm{Aut}(V)$. It induces $\rho: \mathfrak{g} \to \mathrm{End}(V)$.
(by differentiating at the identity: $T_1 G = \mathfrak{g}$, $T_1 \mathrm{Aut}(V) = \mathrm{End}(V)$.)

Prop For $G = U(1) = SO(2)$ (circle group)

choose basis elt $X \in \mathfrak{g}$ such that if ρ is the standard rep of $SO(2)$, then $\rho(X) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Then have $\Theta_G = \alpha \cdot X$ with α G-invt, $\oint_G \alpha = 2\pi$.

Pf Let $g_t = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. Then $X = \frac{\partial}{\partial t}$. So $\Theta_G \big|_{t=0} = dt$. Thus $\Theta_G = dt$.

But then $\oint_G \Theta_G = 2\pi$.