Gauss-Bonnet Thm

M oriented Riemannian 2-manifold

P = bundle of oriented frames

P is a principal $SO(2)$-bundle over M.

Fix a vector field $Y \in TM$ with isolated zeroes. $M' = M \setminus \{\text{zeroes of } Y\}$.

Let $u = Y/\|Y\|$. u determines a unique oriented ON basis $\{u, v\}$.

$\Rightarrow$ u is a section of P over M'.

Say $Y(m) = 0$. Fix a nbhd U of m, and a section s of $P|_U$.

Then on $U \setminus \{m\}$, $u = sg$ for some $g : U \setminus \{m\} \to SO(2)$

Fix an $S^1_\varepsilon \subset U \setminus \{m\}$, then have $g : S^1_\varepsilon \to SO(2)$.

<u>Def</u> The <u>index</u> of Y at m is the <u>degree</u> of g, i.e.

$$\operatorname{ind}_m Y = \frac{1}{2\pi} \oint_{S^1_\varepsilon} g^* \alpha. \qquad [\theta_G = \alpha \cdot X]$$

<u>Lemma</u> Fix any conn. ω in P. Then $[\operatorname{ind}_m Y] \cdot X = \dfrac{1}{2\pi} \lim\limits_{\varepsilon \to 0} \oint_{S^1_\varepsilon} u^* \omega$.

<u>Pf</u> In the local triv given by s, $P|_U \simeq U \times G$

$\omega = A + \theta_G$ where A annihilates vertical vectors

$$\text{Thus } \lim_{\varepsilon \to 0} \oint_{S^1_\varepsilon} u^* \omega = \lim_{\varepsilon \to 0} \oint_{S^1_\varepsilon} g^*(A + \theta_G) = \oint_{S^1_\varepsilon} g^* \theta_G \qquad (\text{contrib. from } A \text{ vanishes as } \varepsilon \to 0)$$

<u>Thm</u> (Gauss-Bonnet) $\displaystyle\sum_{m : Y(m) = 0} \operatorname{ind}_m Y = \frac{1}{4\pi} \int_M S \cdot \mathrm{vol}.$ $\qquad$ (S = scalar curv.)

<u>Pf</u> P carries Levi-Civita connection.

[X = generator of $\mathfrak{g}$ as before]

Then, $\Omega = \pi^*(F)$

and F has $F(e_1, e_2) e_1 = R_{1212} e_2$ i.e. $F(e_1, e_2) = R_{1212} X = (\tfrac{1}{2} S) X$

ie $F = (\tfrac{1}{2} S)\, \mathrm{vol} \cdot X$

Let $M_\varepsilon = M \setminus (\varepsilon\text{-balls around each } m \text{ with } Y(m)=0)$.

Then $$\frac{1}{2}\int_{M_\varepsilon} S \cdot \mathrm{vol} \cdot X = \frac{1}{2}\int_{M_\varepsilon} u^* \pi^* (S \cdot \mathrm{vol}) \cdot X = \int_{M_\varepsilon} u^* \Omega = \int_{u(M_\varepsilon)} \Omega = \oint_{\partial M_\varepsilon} \omega$$

take limit $\varepsilon \to 0$, this becomes

ie $\frac{1}{2}\int S \cdot \mathrm{vol} = 2\pi \sum \mathrm{ind}_m Y$

by previous Lemma.