Special Holonomy

First, fill in a bit more about <u>holonomy in principal bundles</u>

M smooth mfd, P principal G-bundle over M, ω connection in P.

<u>Def/Prop</u> Given a path $\gamma : [0,1] \to M$ and $p \in P_{\gamma(0)}$, the <u>lift</u> $\tilde{\gamma}_p : [0,1] \to P$ is the unique map with $\tilde{\gamma}_p(0) = p$, $\tilde{\gamma}_p'(t) \in \ker \omega \ \forall t$.

<u>Def</u>
$$\mathbb{P}_{\omega,\gamma} : P_{\gamma(0)} \to P_{\gamma(1)}$$
$$p \mapsto \tilde{\gamma}_p(1)$$

<u>Prop</u> $\mathbb{P}_{\omega,\gamma}(p)g = \mathbb{P}_{\omega,\gamma}(pg)$.

<u>Pf</u> $\tilde{\gamma}_{pg}(t) = \tilde{\gamma}_p(t)g$, since it indeed has $\tilde{\gamma}_{pg}(0) = pg$ and $\tilde{\gamma}_{pg}'(t) = g_* \tilde{\gamma}_p'(t) \in g_* \ker \omega$
$= \ker \omega$.

<u>Def</u> $\mathrm{Aut}\, P_x = \{ \varphi : P_x \to P_x \text{ s.t. } \varphi(p)g = \varphi(pg) \}$.
$\mathrm{Aut}\, P = \coprod_x \mathrm{Aut}\, P_x$

<u>Prop</u> $\mathrm{Aut}\, P = P \times_G G$ where G acts on itself by <u>adjoint</u> action, $\mathrm{Ad}_g(g') = g g' g^{-1}$

<u>Pf</u> $\varphi \mapsto \{ (p, g) : \varphi(p) = pg \}$

<u>well defined</u> b/c $\varphi(pg') = (pg')(g'^{-1} g g')$

and check it's bijective..

<u>Rk</u> In any local triv of P, $\mathrm{Aut}\, P$ just acts as <u>left multiplication</u> by G.

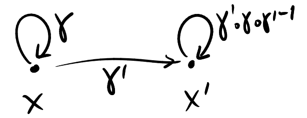
<u>Def</u> $L_x M = \{ \gamma : [0,1] \to M, \ \gamma(0) = \gamma(1) = x \}$

<u>Cor</u> $\gamma \in L_x M \Rightarrow \mathbb{P}_{\omega,\gamma} \in \mathrm{Aut}\, P_x$

<u>Def</u> $\mathrm{Hol}_x \omega = \{ \mathbb{P}_{\omega,\gamma} : \gamma \in L_x M \} \subset \mathrm{Aut}\, P_x$

$\mathrm{Hol}^\circ_x \omega = \{ \mathbb{P}_{\omega,\gamma} : \gamma \in L_x M, \ \gamma \text{ homotopic to trivial loop} \} \subset \mathrm{Aut}\, P_x$.

Prop M connected $\Rightarrow \mathrm{Hol}_x\omega \simeq \mathrm{Hol}_{x'}\omega \quad \forall x, x' \in M$.

Pf  $\mathbb{P}_{\gamma'\gamma\gamma'^{-1}} = \mathbb{P}_{\gamma'} \circ \mathbb{P}_{\gamma} \circ \mathbb{P}_{\gamma'^{-1}}$ so $\underset{\mathrm{Hol}_x\omega}{\underset{\cap}{g}} \mapsto \mathbb{P}_{\gamma'} \circ g \circ \mathbb{P}_{\gamma'^{-1}}$ does the job, with $\mathbb{P}_{\gamma'} \circ g \circ \mathbb{P}_{\gamma'^{-1}} \in \mathrm{Hol}_{x'}\omega$.

Alternatively: **Def** For $p \in P$, $\mathrm{Hol}_p\omega = \{g : pg$ can be reached from p by parallel transport$\} \subset G$

Prop M connected $\Rightarrow$ all $\mathrm{Hol}_p\omega$ and all $\mathrm{Hol}_x\omega$ are <u>isomorphic</u>.

Pf Exercise.

If $\mathrm{Hol}_x\omega \subsetneq \mathrm{Aut}\, P_x$, then P can be reduced to a smaller structure group:

Thm Let $H = \mathrm{Hol}_p\omega$. Define $Q = \{q \in P \mid q$ can be reached from p by parallel transport$\}$
Then Q is a principal H-bundle and $P = Q \times_H G$.

Pf H acts on Q by $q = \mathbb{P}_{\tilde{\gamma},\omega}(p) \mapsto qh = \mathbb{P}_{\tilde{\gamma},\omega}(h^{-1}(p))$.
(Well defined since if $\mathbb{P}_{\tilde{\gamma},\omega}(p) = \mathbb{P}_{\tilde{\gamma}',\omega}(p)$ then $\mathbb{P}_{\tilde{\gamma},\omega} = \mathbb{P}_{\tilde{\gamma}',\omega}$ so in ptic $\mathbb{P}_{\tilde{\gamma},\omega}(h^{-1}p) = \mathbb{P}_{\tilde{\gamma}',\omega}(h^{-1}p)$)
and $Q/H = M$ [Exercise].

The desired map $Q \times_H G \xrightarrow{\sim} P$
is $(\mathbb{P}_{\tilde{\gamma},\omega}(p), g) \mapsto \mathbb{P}_{\tilde{\gamma},\omega}(pg)$.

Ex For M Riemannian and P = bundle of frames, fix some $e \in P_x$.
Then Q = all frames related to e by parallel transport.
If e is an orthogonal frame, $\mathrm{Hol}_e\omega \subset O(n)$.
So P can be reduced to $O(n)$, <u>or smaller</u>. [We already knew this, but now see it as a conseq. of the holonomy gp.]

[**Rk** We didn't just get an abstract <u>reduction</u> but something more concrete:
Def A <u>G-structure</u> on M is a principal G-subbundle $Q \subset P$.]

What <u>other</u> reductions can we get in this way?

The possibilities have been classified:

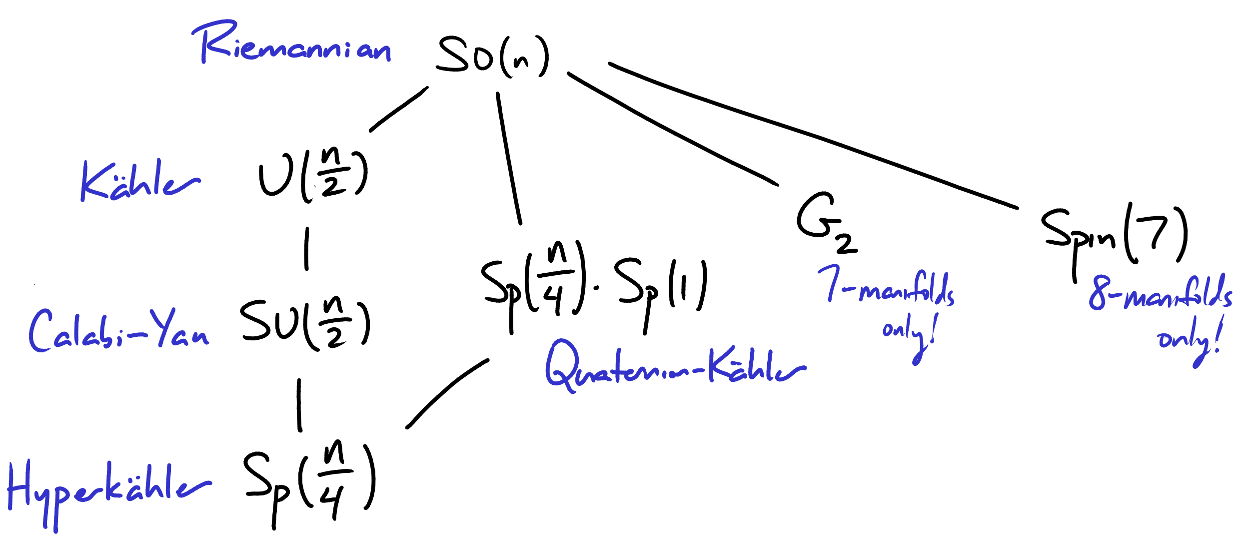
<u>Def</u> M is a <u>Riemannian symmetric space</u> if $\forall x \in M$, $\exists g_x \in \mathrm{Isom}(M)$ s.t. $g_x(x) = x$, $g_{x_*} = -1$ on $T_x M$.

[<u>Prop</u> M is Riem sym $\iff$ M is homogeneous and $\exists x \in M$, $g_x \in \mathrm{Isom}(M)$ s.t. $g_{x_*}(x) = x$, $g_{x_*} = -1$ on $T_x M$.]

<u>Def</u> M is <u>locally Riemannian symmetric space</u> if $\forall x \in M$, $\exists$ a nbhd of x which is isometric to an open subset of a Riem symmetric space.

<u>Def</u> M is <u>reducible</u> if $M \simeq (M_1 \times M_2)/\Gamma$ for some M_1, M_2 and $\Gamma \subset \mathrm{Isom}(M_1 \times M_2)$ finite. <u>Irreducible</u> otherwise.

<u>Thm</u> [Berger]: If M is <u>not locally symmetric</u>, <u>irreducible</u> and <u>simply connected</u>, holonomy gp is $\simeq$ one of the following:



Here
- $Sp(k) = \{ g \in GL(k, \mathbb{H}) : \langle x, y \rangle = \langle gx, gy \rangle \}$ with $\langle , \rangle$ the standard pairing on $\mathbb{H}^k$

  e.g. $Sp(1) \simeq SU(2)$

- $G_2 = \{\text{automorphisms of } \mathbb{O}\}$; $\dim_{\mathbb{R}} G_2 = 14$; G_2 has 7-dim irrep.