Lecture 1 | 27 Aug 2015

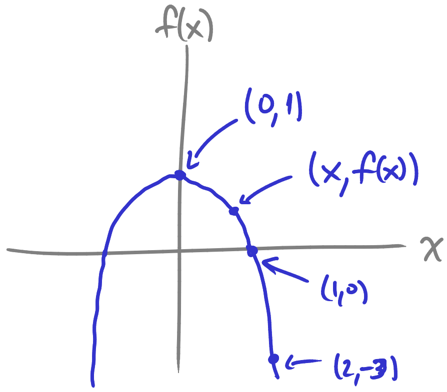
Functions

A <u>function</u> f is a machine which takes an <u>input</u> x and produces an <u>output</u> $f(x)$

For this course. x and $f(x)$ will be (real) <u>numbers</u>.

<u>Ex</u> $f(x) = 1 - x^2$

x	$f(x)$
0	1
1	0
2	-3



The <u>domain</u> of f is the set of all allowed x (x for which $f(x)$ exists.)

<u>Ex</u> for $f(x) = 1-x^2$; domain = all real x
$= \{x : -\infty < x < \infty\}$
$= (-\infty, \infty)$

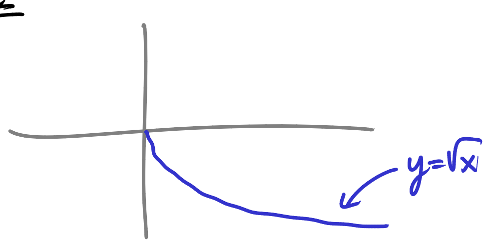
The <u>range</u> of f is the set of all y such that $y = f(x)$ for <u>some</u> x.

<u>Ex</u> $f(x) = 1-x^2$ has range $(-\infty, 1] = \{y : -\infty < y \leq 1\}$

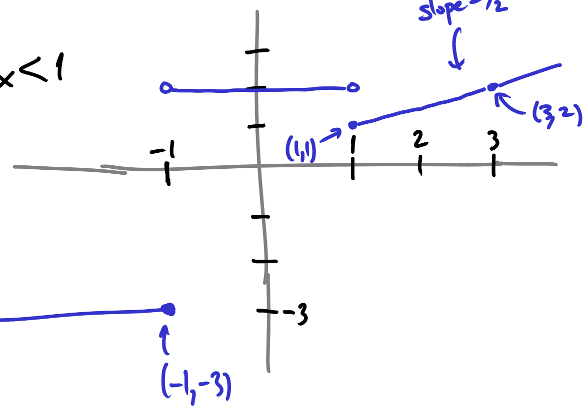
<u>Ex</u> $f(x) = -\sqrt{x}$: domain $= [0, \infty) = \{x : 0 \leq x < \infty\}$

range $= (-\infty, 0] = \{y : -\infty < y \leq 0\}$

[Why? This means: for any $y \in (-\infty, 0]$ there is some x such that $y = -\sqrt{x}$.
To find this x, square both sides: $y^2 = (-\sqrt{x})^2 = x$
so $\underline{\underline{x = y^2}}$]



Ex $f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{if } x \geq 1 \\ 2 & \text{if } -1 < x < 1 \\ -3 & \text{if } x \leq -1 \end{cases}$



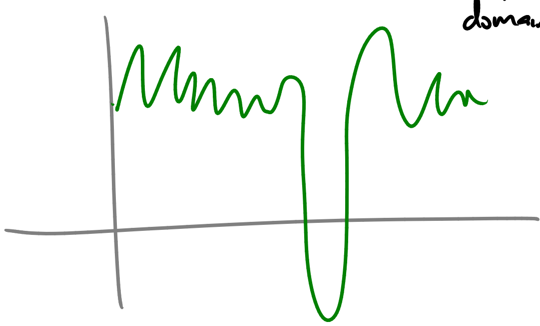
domain $= (-\infty, \infty)$

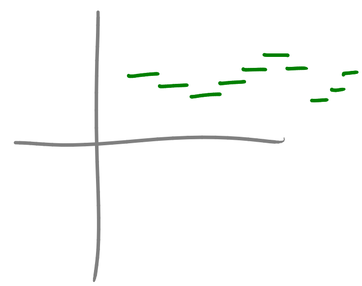
range $= \{-3\} \cup [1, \infty)$

$= \{y : y = -3 \text{ or } y \geq 1\}$

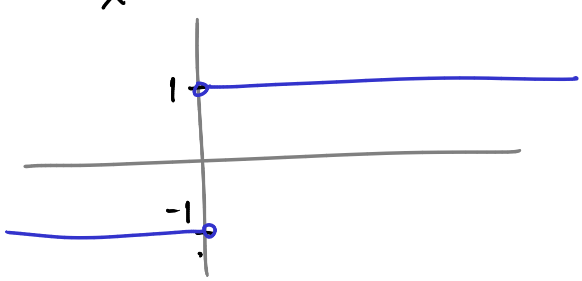
Ex $f(x) =$ value of DJIA at x seconds past opening today

domain $= [0, \#\text{sec open in a day}]$





Ex $f(x) = \dfrac{|x|}{x}$ domain $= \{x \neq 0\} = (-\infty, 0) \cup (0, \infty)$



x	$f(x)$
1	$1/1 = 1$
2	$2/2 = 1$
3	$3/3 = 1$
-1	$\frac{\lvert -1 \rvert}{-1} = \frac{1}{-1} = -1$
-2	$\frac{\lvert -2 \rvert}{-2} = \frac{2}{-2} = -1$
$\vdots$	

if $x > 0$, $f(x) = \dfrac{|x|}{x} = \dfrac{x}{x} = 1$

if $x < 0$, $f(x) = \dfrac{|x|}{x} = \dfrac{-x}{x} = -1$

so f could also be written $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

A function f is called <u>even</u> if for all x in the domain,

$$f(-x) = f(x)$$

f is called <u>odd</u> if for all x in the domain,

$$f(-x) = -f(x)$$

Ex $f(x) = \dfrac{2+x^2}{x^6}$ $\quad f(-x) = \dfrac{2+(-x)^2}{(-x)^6} = \dfrac{2+x^2}{x^6} = f(x)$

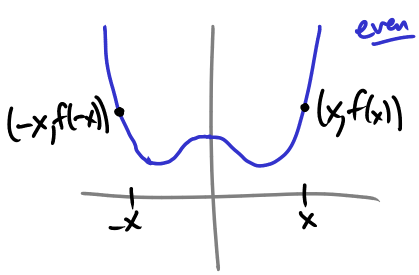
$\longrightarrow$ f is even

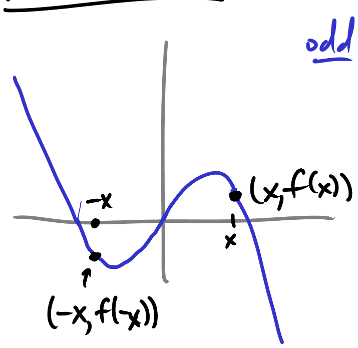
$f(x) = \dfrac{-x}{1+x^2}$ $\quad f(-x) = \dfrac{-(-x)}{1+(-x)^2} = \dfrac{x}{1+x^2} = -f(x)$

$\longrightarrow$ f is odd

$f(x) = x+x^2$ $\quad f(-x) = -x+(-x)^2 = -x+x^2$

$\longrightarrow$ f not even or odd





Classes of functions

Linear $\quad f(x) = ax+b$ $\qquad$ ex $f(x) = 7x-4$

Polynomial $\quad f(x) = ax^n + bx^{n-1} + \dots + z$ $\qquad$ ex $f(x) = -2x^4+3x^3+5x+8$

Rational $\quad f(x) = \dfrac{P(x)}{Q(x)}$ $\quad$ P, Q polynomial $\qquad$ ex $f(x) = \dfrac{3+7x+2x^2}{8-x^{1001}}$

Power $\quad f(x) = x^a$

ex $f(x) = x^2$

$f(x) = x^{1/2} = \sqrt{x}$

$f(x) = x^{1/7} = \sqrt[7]{x}$

$f(x) = x^{-3} = \dfrac{1}{x^3}$

$f(x) = x^{2/5} = \sqrt[5]{x^2} = \left(\sqrt[5]{x}\right)^2$

Exponential functions $\quad f(x) = a^x$ $\qquad$ a = constant, $a > 0$

ex $f(x) = 2^x$

$f(x) = \pi^x$

What does a^x mean?

If $x = \frac{p}{q}$ p, q integers then $a^x = a^{p/q} = (a^p)^{\frac{1}{q}} = \sqrt[q]{a^p}$

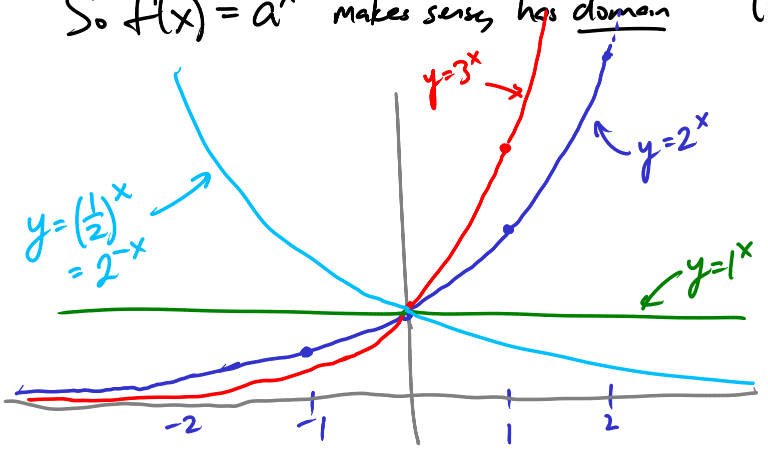
But what if e.g. $x = \pi$? What is 2^π?

<u>Fact</u>: the def^n of a^x <u>can</u> be extended to $x =$ any real number!

e.g. 2^π makes sense.

$$\left[\begin{array}{l} 2^3 < 2^{3.1} < 2^{3.14} < 2^{3.141} < 2^{3.1415} < 2^{3.14159} < \cdots < 2^\pi \\ 2^4 > 2^{3.2} > 2^{3.15} > 2^{3.142} > 2^{3.1416} \geq 2^{3.14160} > \cdots > 2^\pi \\ 2^\pi \text{ is the } \underline{\underline{\text{unique}}} \text{ \# obeying these constraints} \end{array}\right]$$

So $f(x) = a^x$ makes sense, has <u>domain</u> $(-\infty, \infty)$



x	2^x
0	1
1	2
2	4
3	8
-1	1/2
-2	1/4
⋮	⋮

NB: graph of $y = \left(\frac{1}{2}\right)^x = 2^{-x}$ is obtained by <u>reflecting</u> graph of $y = 2^x$ in y-axis.

similarly for any $f(x) \longleftrightarrow f(-x)$

<u>Laws of exponents</u>

① $a^{x+y} = a^x a^y$

② $a^{-x} = \frac{1}{a^x}$

③ $(a^x)^y = a^{xy}$

④ $(ab)^x = a^x b^x$

<u>Ex</u> $\frac{1}{a^3} = a^{-3}$

$9 \cdot 3^x = 3^2 \cdot 3^x = 3^{x+2}$

Fact If $a \neq 1$ and $a^x = a^y$ then $x = y$.

(the function $f(x) = a^x$ is "1-1")

Ex If $5^x = 25^{3x-2}$ what is x?

$$5^x = (5^2)^{3x-2}$$

$$5^x = 5^{6x-4}$$

$$x = 6x - 4$$

$$-5x = -4$$

$$\underline{x = 4/5}$$

Ex If $f(x) = x^2$ and h is any real #

what is $\dfrac{f(x+h) - f(x)}{h}$?

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \frac{2xh + h^2}{h} = \underline{2x + h}$$