Lecture 2 | 1 Sep 2015

Composition of functions

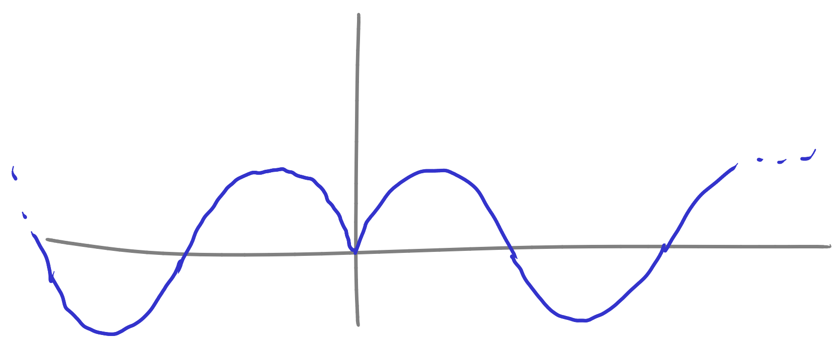
If f, g are functions, with range$(g) \subset$ domain(f), then we can define a new function $f \circ g$ by the formula $f \circ g(x) = f(g(x))$

Ex $f(x) = x^2 + 1$
$g(x) = 6x$

then $f(g(x)) = f(6x) = (6x)^2 + 1 = 36x^2 + 1$

Ex $f(x) = \sin x$
$g(x) = |x|$

$(f \circ g)(x) = f(|x|) = \sin |x|$



even function: $\sin |x| = \sin |-x|$

Inverse functions

If f is a function, its inverse is a function f^{-1} such that $(f^{-1} \circ f)(x) = x$

domain(f^{-1}) = range(f)
range(f^{-1}) = domain(f)

$$x \rightarrow \boxed{f} \rightarrow f(x) \rightarrow \boxed{f^{-1}} \rightarrow x$$

x	$f(x)$
1	7
2	11
3	18

x	$f^{-1}(x)$
7	1
11	2
18	3

Ex If $f(x) = 13x$ then $f^{-1}(x) = \frac{x}{13}$

(Because $f^{-1}(f(x)) = f^{-1}(13x) = \frac{13x}{13} = x$)

Ex If $f(x) = x+3$ then $f^{-1}(x) = x-3$

Some functions f don't have inverses:

e.g.

x	$f(x)$
1	8
2	8
3	2

x	$f^{-1}(x)$
8	1
8	2
2	3

← no such function f^{-1}!

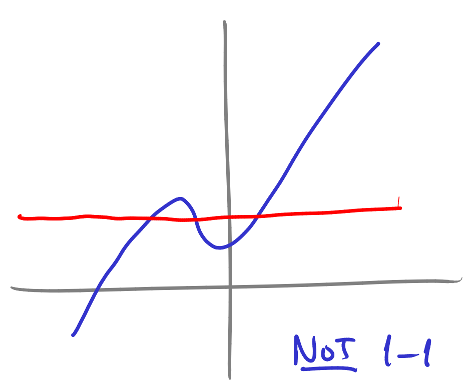
We say f is a 1-1 function if

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2 .$$

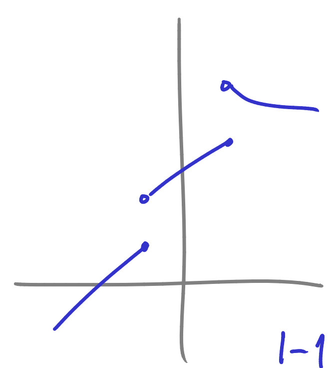
Fact: If f is 1-1, then f^{-1} exists.

Horizontal line test:

f is 1-1 just if no horizontal line meets the graph of f more than once.

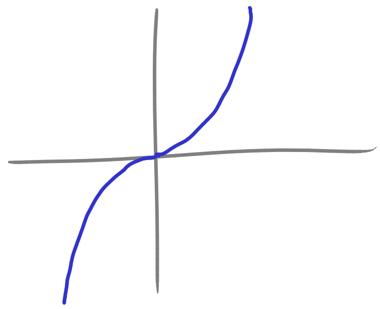


NOT 1-1



1-1

Ex $f(x) = x^3$

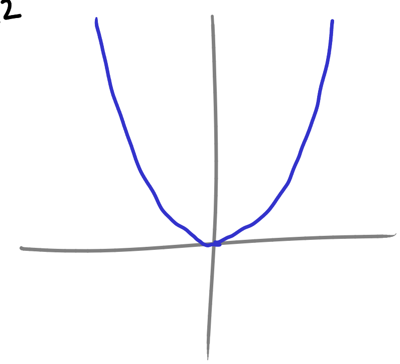


is 1-1.

Its inverse is $f^{-1}(x) = \sqrt[3]{x}$.

(Because $\sqrt[3]{x^3} = x$, for <u>all real</u> x.)

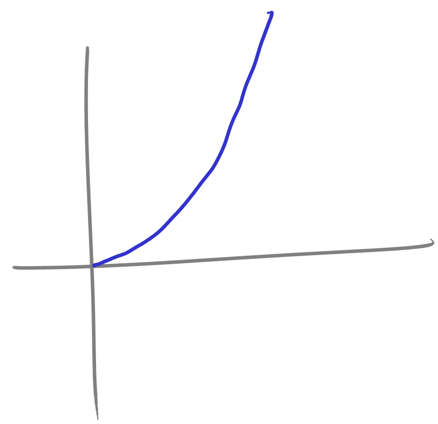
Ex $f(x) = x^2$



<u>not</u> 1-1 on its "natural domain" $(-\infty, \infty)$.

<u>But</u>, if we take the domain $[0, \infty)$

then f <u>is</u> 1-1 on this domain.



$\Longrightarrow$ it has an inverse $f^{-1}(x) = \sqrt{x}$

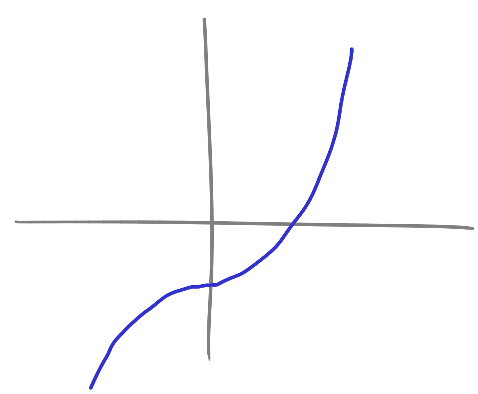
(because $\sqrt{x^2} = x$ <u>if</u> $x \in [0, \infty)$)

Ex $f(x) = x^3 - 1$

To find $f^{-1}(x)$: <u>solve for x</u> in this equation

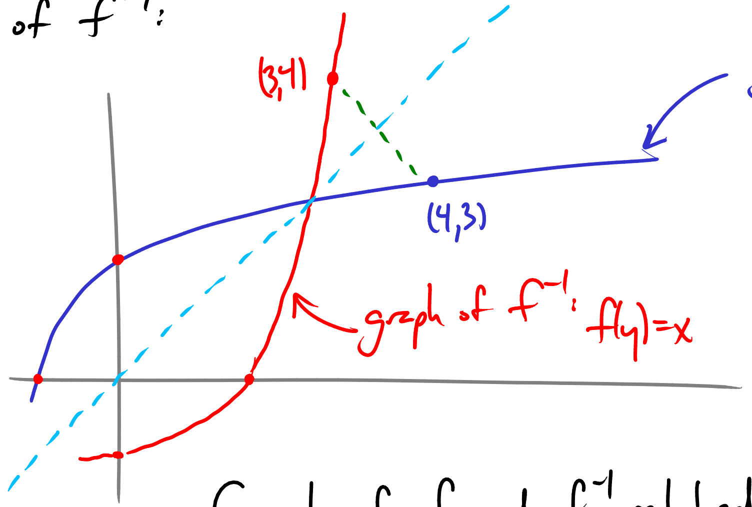
ie let $y = f(x)$, then

$$y = x^3 - 1$$
$$y + 1 = x^3$$
$$x = \sqrt[3]{y+1}$$



thus $f^{-1}(y) = \sqrt[3]{y+1}$ (or, $f^{-1}(x) = \sqrt[3]{x+1}$)

A picture of f^{-1}:



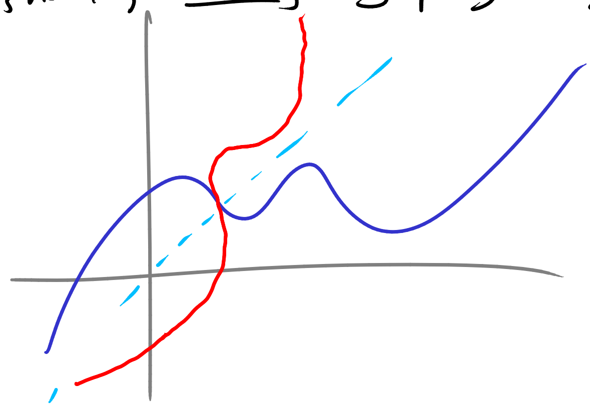
graph of f $\quad y=f(x)$

graph of f^{-1}: $\quad y=f^{-1}(x)$

$f(y)=x$

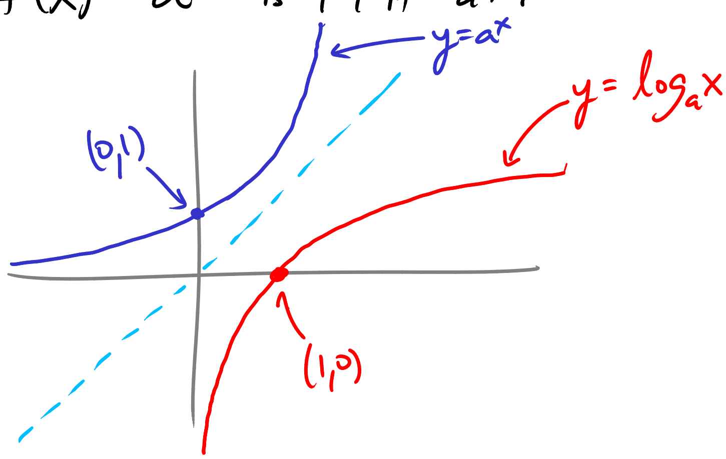
Graphs of f and f^{-1} related by "swapping $x \leftrightarrow y$"
or equivalently "reflection in the axis $y=x$"

[If f is not 1-1, <u>reflecting</u> its graph gives a graph that doesn't obey vertical line test



— so not the graph of an actual <u>function</u>]

<u>Ex</u> $f(x)=a^x$ is 1-1 if $a \neq 1$



We call its inverse

$$f^{-1}(x)=\log_a x$$

domain: $(0,\infty)$

x	10^x
0	1
1	10
2	100
3	1000
4	10,000
−1	0.1
−2	0.01

x	$\log_{10} x$
1	0
10	1
100	2
1000	3
10 000	4
0.1	−1
0.01	−2

Laws of logarithms

① $\log_a(xy) = \log_a x + \log_a y$

② $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

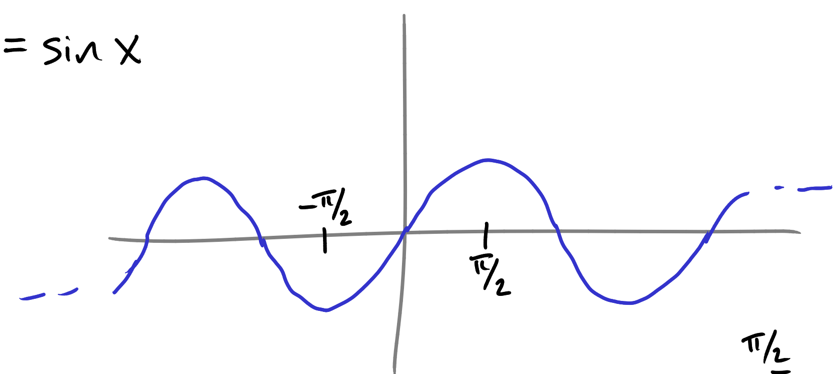
③ $\log_a(x^r) = r\log_a x$

[Why? eg, ① comes from $a^x a^y = a^{x+y}$]

Ex $\log_3 63 - \log_3 7 = \log_3\left(\frac{63}{7}\right) = \log_3(9) = 2$ (because $3^2 = 9$)

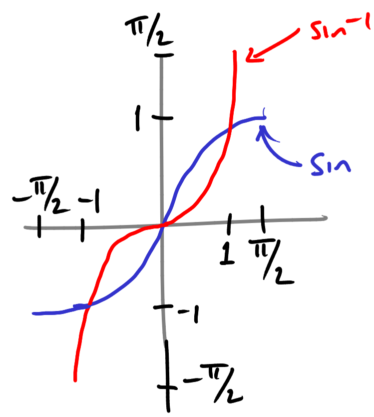
$\left[\log_3(3^2) = 2 \cdot \log_3 3 = 2 \cdot 1 = 2\right]$

Ex $f(x) = \sin x$



not 1-1 on domain $(-\infty, \infty)$

but is 1-1 on domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Let $\sin^{-1}$ mean the inverse of sin on domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

So, in practice, "$\sin^{-1}x$" means "an angle θ with $\sin\theta = x$ such that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$"

Ex $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$, because $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $-\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2}$.

Similarly: (1) $f(x) = \cos x$ is 1-1 on domain $[0, \pi]$

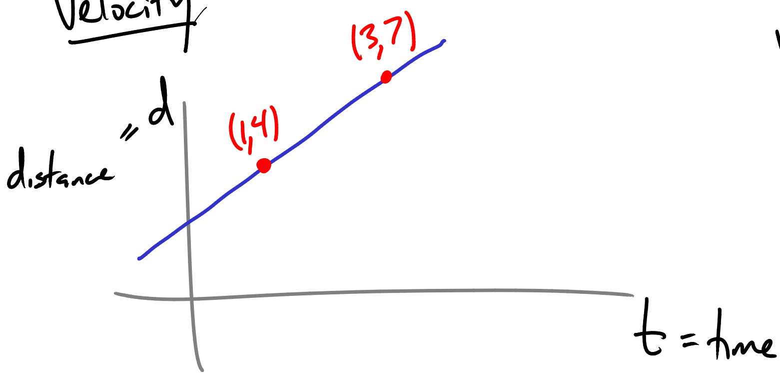
let $\cos^{-1} x$ be the inverse of the f^n with that domain

(2) $f(x) = \tan x$ is 1-1 on domain $(-\pi/2, \pi/2)$ ---

$$f(x) = x^2$$

$$y = x^2 \qquad\qquad x = \pm\sqrt{y}$$

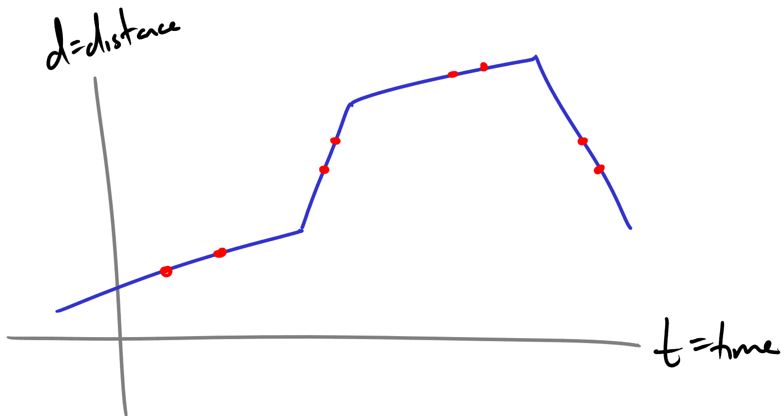
Velocity



velocity is the slope of this graph

$$= \frac{7-4 \text{ miles}}{3-1 \text{ minutes}} = \frac{3 \text{ mi}}{2 \text{ min}} = 1.5 \ \frac{\text{mi}}{\text{min}}$$

$(= 90 \text{ mph})$



Now have different speeds for different segments of the trip: each still given by slope of the line segment