Lecture 4

8 Sep 2015

Today: my office hr at 1pm (1-2)

Last time: limits and limit laws ①-⑧

⑦ $\lim_{x \to a} c = c$

⑧ $\lim_{x \to a} x = a$

⑨ $\lim_{x \to a} x^n = a^n$

Could prove this using the Limit Laws we already know: e.g.

$$\lim_{x \to a} x^2 = \lim_{x \to a} (x \cdot x) = \left(\lim_{x \to a} x\right) \cdot \left(\lim_{x \to a} x\right) = a \cdot a = a^2$$

⑩ $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$

⑪ $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

Ex $\lim_{x \to 0} x^2 + 9 = 0^2 + 9 = 9$

for polynomials or rational functions, get limits by just plugging in

$$\lim_{x \to 0} \sqrt{x^2 + 9} = \sqrt{\lim_{x \to 0} x^2 + 9} = \sqrt{9} = 3$$

by Limit Law 11

Q: what about one-sided limits?

A. there are Limit Laws for those too. e.g. $\lim_{x \to a^+} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a^+} f(x)}$

Ex $\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$ naive subst. gives $\frac{0}{0}$:

so need some clever way of simplifying

multiply by $\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}$:

then get $\frac{\sqrt{x^2+9}-3}{x^2} \cdot \frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3} = \frac{\left(\sqrt{x^2+9}\right)^2 - 3^2}{x^2\left(\sqrt{x^2+9}+3\right)}$

$$= \frac{x^2+9-9}{x^2\left(\sqrt{x^2+9}+3\right)} = \frac{x^2}{x^2\left(\sqrt{x^2+9}+3\right)}$$

$$= \frac{1}{\sqrt{x^2+9}+3}$$

and now $\lim_{x \to 0} \frac{1}{\sqrt{x^2+9}+3} = \frac{1}{6}$

Ex $\lim_{x \to 0} \frac{x+1}{x-2} + \frac{x}{\sin x} = \left(\lim_{x \to 0} \frac{x+1}{x-2}\right) + \left(\lim_{x \to 0} \frac{x}{\sin x}\right)$

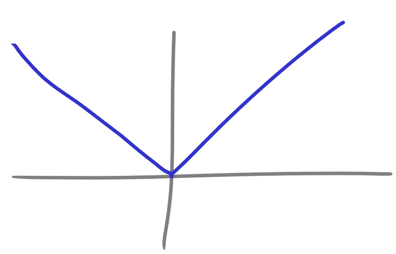
$$= -\frac{1}{2} + \lim_{x \to 0} \frac{1}{\left(\frac{\sin x}{x}\right)}$$

$$= -\frac{1}{2} + \frac{1}{\lim_{x \to 0} \frac{\sin x}{x}}$$

$$= -\frac{1}{2} + \frac{1}{1} = \underline{\underline{\frac{1}{2}}}$$

Now a few that we don't do by Limit Laws:

Ex $\lim_{x \to 0} |x| = ?$



$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = -\lim_{x\to 0^-} x = -0 = 0$$

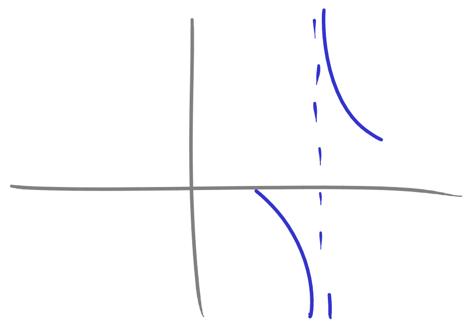
So, $\lim_{x\to 0} |x| = 0$.

Ex $\lim_{x\to 3} \frac{x+2}{x-3}$: can't do by direct subst.

plug in x very close to 3:

then $\frac{x+2}{x-3} = \frac{(\text{close to } 5)}{(\text{very small, } {+ve \text{ if } x>3 \atop -ve \text{ if } x<3})}$

$\Rightarrow$ limit does not exist



Ex $\lim_{x\to 0} x\sin(x) = 0$

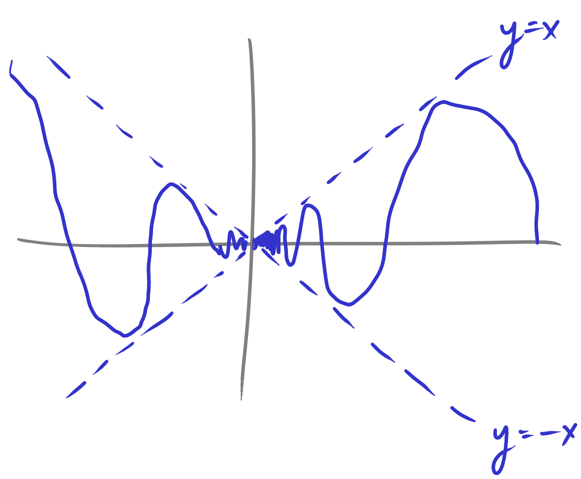
Ex $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

plug in $x = 0.00001$.

$(0.00001)\sin\left(\frac{1}{0.00001}\right)$

this is small!

$\rightarrow \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$



Fact ("Squeeze Theorem"): if $g(x) \le f(x) \le h(x)$ for all x (in the domain of f), and $\lim_{x\to a} g(x) = L$

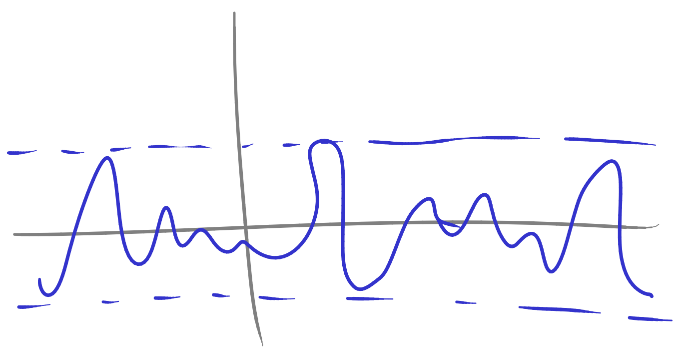
and $\lim_{x\to a} h(x) = L$

then $\lim_{x\to a} f(x) = L$.

Ex when $f(x) = x\sin\left(\frac{1}{x}\right)$, $-|x| \le f(x) \le |x|$ for all x

and we know $\lim_{x\to 0} |x| = 0$ $\quad \lim_{x\to 0} -|x| = 0$

so, $\lim_{x\to 0} f(x) = 0$.



What is a limit, precisely?

We say $\lim_{x\to a} f(x) = L$ if:

for every $\varepsilon > 0$ (no matter how small),

there is some $\delta > 0$

such that whenever $|x-a| < \delta$ (and $x \neq a$)

we have $|f(x) - L| < \varepsilon$.

To see what this means: let's consider the function $f(x) = 5x$.

$\lim_{x \to 1} f(x) = 5$: how does that fit with the definition above?

Say $\varepsilon = 0.1$. Then we want $|f(x) - L| < \varepsilon$

ie $|f(x) - 5| < 0.1$

for $|x-1| < \delta$. What should δ be?

x	$f(x)$	$\lvert f(x)-5 \rvert$
1.1	5.5	0.5
1.01	5.05	0.05
1.001	5.005	0.005
$\vdots$		
0.999	4.995	0.005
0.99	4.95	0.05
0.9	4.5	0.5

If $|x-1| < 0.02$ (or < 0.01)

then $|f(x) - 5| < 0.1$!

OK. So can take $\delta = 0.02$ (or 0.01) when $\varepsilon = 0.1$.

What if $\varepsilon = 0.01$?

$\delta = 0.002$ will work.

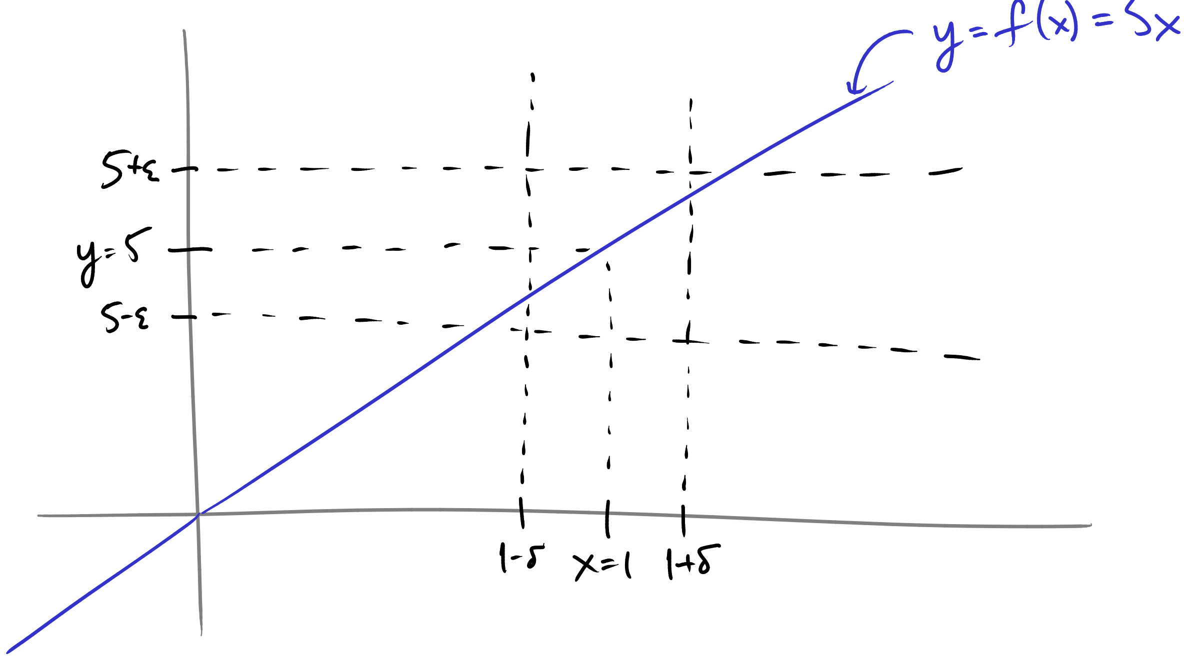
(or 0.001, or anything smaller)

What if $\varepsilon = 0.00000000001$?

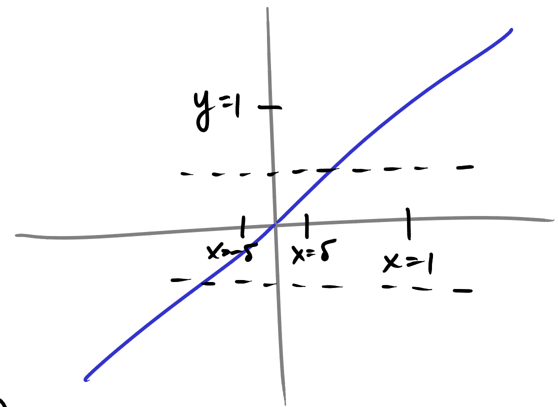
Can we find the δ such that if x is within δ of 1

then $f(x)$ is within $\varepsilon = 0.00000000001$ of 5?

Yes: take $\delta = 0.000000000002$ ie $\delta = \varepsilon/5$



Ex $f(x) = x + \frac{1}{1000000}\sin\left(\frac{1}{x}\right)$



The limit as $x \to 0$

$\lim_{x\to 0} f(x)$ does not exist.

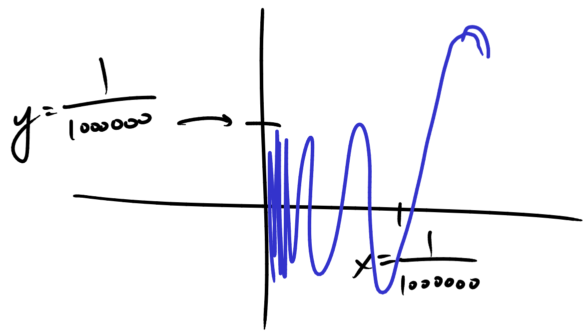
Why? Suppose we thought $\lim_{x\to 0} f(x) = 0$.

If we take $\varepsilon = 0.5$ (y-tolerance)

and $\delta = 0.1$ (x-tolerance)

then for all x with $|x| < \delta$,

we do have $|f(x) - 0| < \varepsilon$.



But, if we take $\varepsilon < \frac{1}{1000000}$ there is no δ that works!