Lecture 4

8 Sep 2015

Reminder: my office hr today 1-2pm.

Last time: limits and "Limit Laws"

Limit Laws cont'd:

(7) $\lim_{x\to a} c = c$ (c any constant)

(8) $\lim_{x\to a} x = a$

(9) $\lim_{x\to a} x^n = a^n$

can get this from the laws we already know:

e.g. $$\lim_{x\to a} x^2 = \lim_{x\to a} (x\cdot x) = \left(\lim_{x\to a} x\right)\cdot\left(\lim_{x\to a} x\right) = a \cdot a = a^2$$

(10) $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$

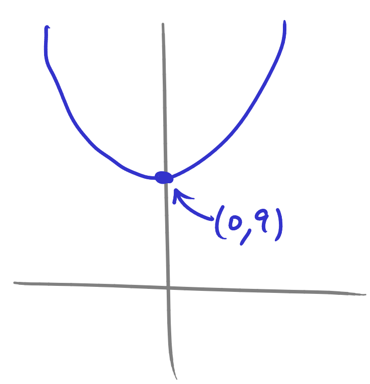
(11) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

<u>Ex</u> $\lim_{x\to 0} x^2+9$

$$= \lim_{x\to 0} x^2 + \lim_{x\to 0} 9$$

$$= 0^2 + 9$$

$$= 9$$



By same kind of reasoning: if $f(x) = \frac{P(x)}{Q(x)}$ rational function

then $\lim_{x \to a} f(x) = \frac{\lim_{x \to a} P(x)}{\lim_{x \to a} Q(x)} = \frac{P(a)}{Q(a)} = f(a)$ (if $Q(a) \neq 0$)

Ex $\lim_{x \to 0} \sqrt{x^2+9} = \sqrt{\lim_{x \to 0} x^2+9} = \sqrt{9} = 3$

use Limit Law 11

Ex $\lim_{x \to 0} \frac{\sqrt{x^2+9} - 3}{x^2}$ plug in x = small #: $\frac{\text{small \#}}{x^2} = \frac{\text{small \#}}{(\text{another small \#})^2}$

inconclusive

$$\frac{\sqrt{x^2+9} - 3}{x^2} \cdot \frac{\sqrt{x^2+9} + 3}{\sqrt{x^2+9} + 3} = \frac{(\sqrt{x^2+9})^2 - 3^2}{x^2(\sqrt{x^2+9} + 3)}$$

$$= \frac{x^2+9-9}{x^2(\sqrt{x^2+9}+3)}$$

$$= \frac{x^2}{x^2(\sqrt{x^2+9}+3)} = \frac{1}{\sqrt{x^2+9}+3}$$

So $\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2} = \frac{1}{\lim_{x \to 0} \sqrt{x^2+9}+3}$

$= \frac{1}{6}$

Ex $\lim_{x \to 0} \frac{x+1}{x-2} + \frac{\sin x}{x} = \lim_{x \to 0} \frac{x+1}{x-2} + \lim_{x \to 0} \frac{x}{\sin x}$

$$= \frac{0+1}{0-2} + \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{-1}$$

$$= -\frac{1}{2} + \left[\lim_{x \to 0} \frac{\sin x}{x}\right]^{-1}$$

$$= -\frac{1}{2} + 1^{-1}$$

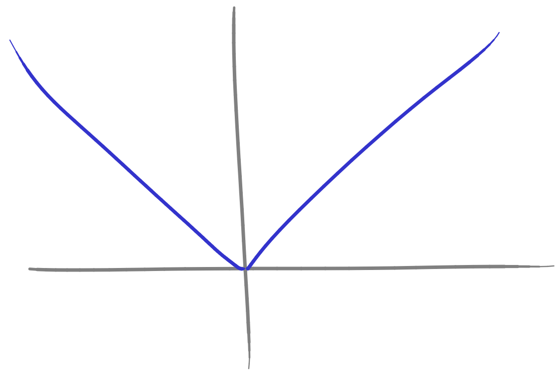
$$= \frac{1}{2}$$

(recall from last lecture $\lim_{x\to 0} \frac{\sin x}{x} = 1$ — should memorize this!)

<u>Fact</u> Limit Laws also work for $\lim_{x\to a^+}$ or $\lim_{x\to a^-}$

for example $\lim_{x\to a^+} f(x)g(x) = \left(\lim_{x\to a^+} f(x)\right)\left(\lim_{x\to a^+} g(x)\right)$

<u>Ex</u> $\lim_{x\to 0} |x| = 0$



by looking at graph, or,

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} -x = -\lim_{x\to 0^-} x = -0 = 0$$

So, $\lim_{x\to 0} |x| = \underline{\underline{0}}$.

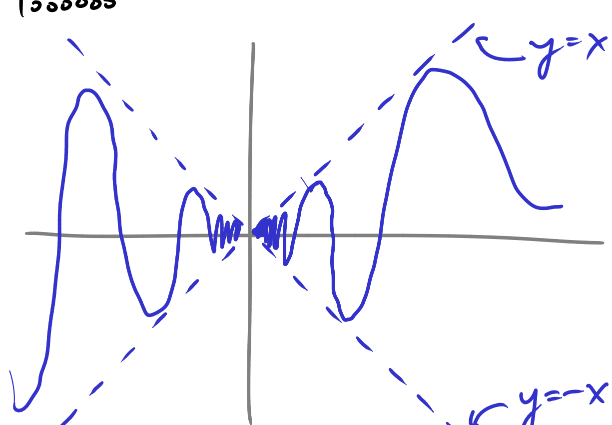
<u>Ex</u> $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

Can't use Limit Law: $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ DNE.

Plug in x small, say $x = \frac{1}{1000}$: $\frac{1}{1000} \cdot \sin(1000)$ — small!

$x = \frac{1}{1000000}$: $\frac{1}{1000000} \sin(1000000)$ — smaller!

Indeed, $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$



__Fact__ ("Squeeze Theorem"): if $g(x) \le f(x) \le h(x)$ for all x in domain of f

and $\lim_{x\to a} g(x) = L$, $\lim_{x\to a} h(x) = L$

then, $\lim_{x\to a} f(x) = L$.

e.g. $-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$

and $\lim_{x\to 0} |x| = 0$, $\lim_{x\to 0} -|x| = 0$, so, $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

A precise definition of __limit__:

We say $\lim_{x\to a} f(x) = L$

if for any $\varepsilon > 0$ (no matter how small) ("y-tolerance")

there exists a $\delta > 0$ ("x-tolerance"),

__such that__

whenever $|x-a| < \delta$ (and $x \ne a$)

we have $|f(x) - L| < \varepsilon$.

__Ex__ $f(x) = 5x$ $\lim_{x\to 1} 5x = 5$. So $a=1$, $L=5$

x	$f(x)$	$\|x-1\|$	$\|f(x)-5\|$
1.1	5.5	0.1	0.5
1.01	5.05	0.01	0.05
1.001	5.005	0.001	0.005
1.0001	5.0005	0.0001	0.0005
.9999	4.9995	0.0001	0.0005
.999	4.995	0.001	0.005
.99	4.95	0.01	0.05
.9	4.5	0.1	0.5

If $\varepsilon = 0.1$

we can take $\delta = 0.01$

— then if $|x-1| < \delta$

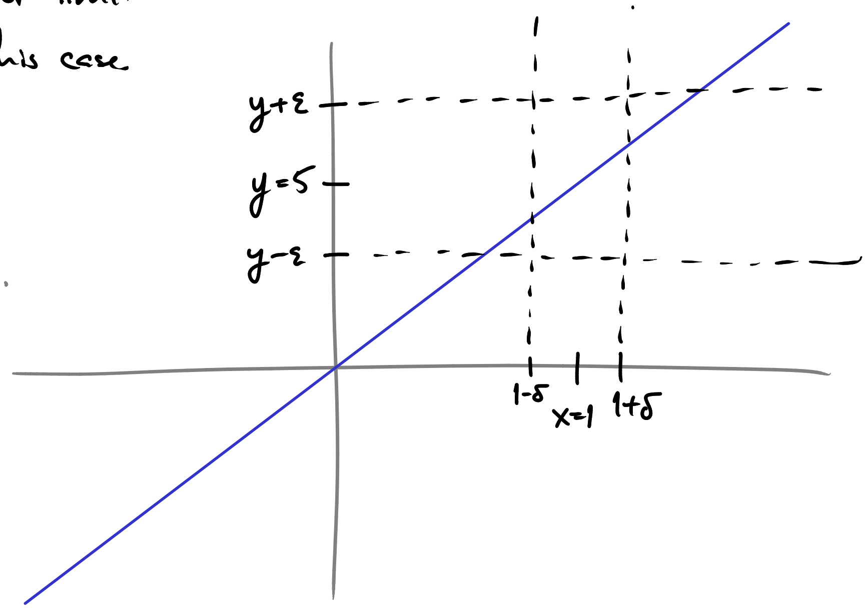
we indeed have $|f(x) - 5| < \varepsilon$

If $\varepsilon = 0.01$

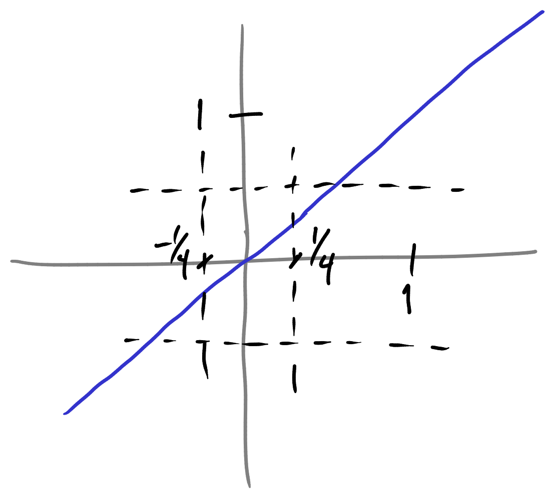
we can take $\delta = 0.001$.

In fact for any $\varepsilon > 0$ (y-tolerance) we can pick δ (x-tolerance) to be $\delta = \frac{\varepsilon}{10}$.

So, the definition of limit **is** satisfied in this case



Ex $\lim_{x \to 0} x + \frac{1}{1000000}\sin\left(\frac{1}{x}\right)$



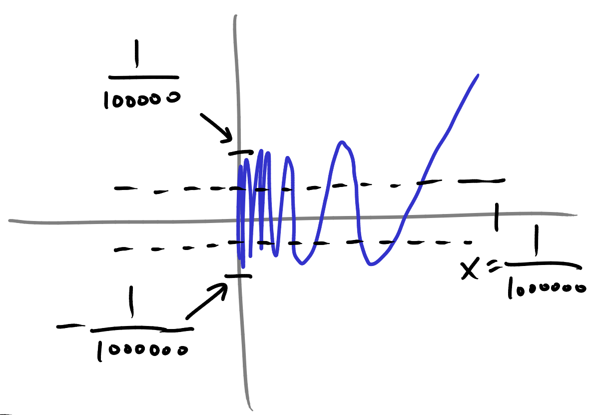
Does this limit exist?

Is it $\lim_{x \to 0} x + \frac{1}{1000000}\sin\left(\frac{1}{x}\right) = 0$?

Given y-tolerance $\varepsilon = \frac{1}{2}$,

we can take x-tolerance $\delta = \frac{1}{4}$.

But what if we take $\varepsilon = \frac{1}{10000000}$?



Then there is **no** δ that will work!

So, $\lim_{x \to 0} x + \frac{1}{1000000}\sin\left(\frac{1}{x}\right)$ DNE.