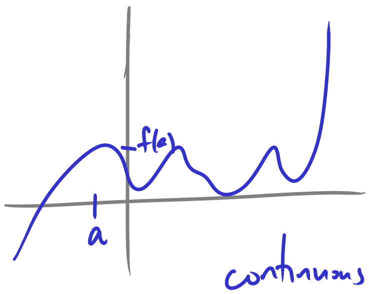
# Lecture 5

10 Sep 2015

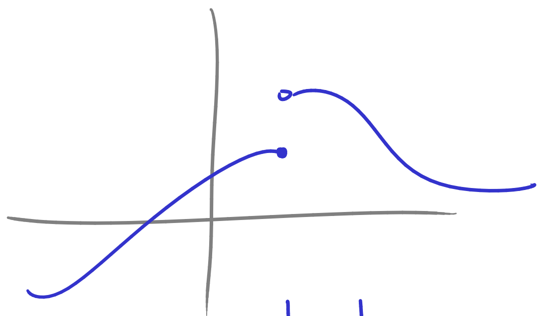
Last time: limits and their formal definition

## Continuity

Informally: we say $f$ is continuous if "we can draw it w/o lifting the pencil"




continuous



not continuous

Formally: we say $f$ is continuous at $a$ if $\lim\limits_{x \to a} f(x) = f(a)$.

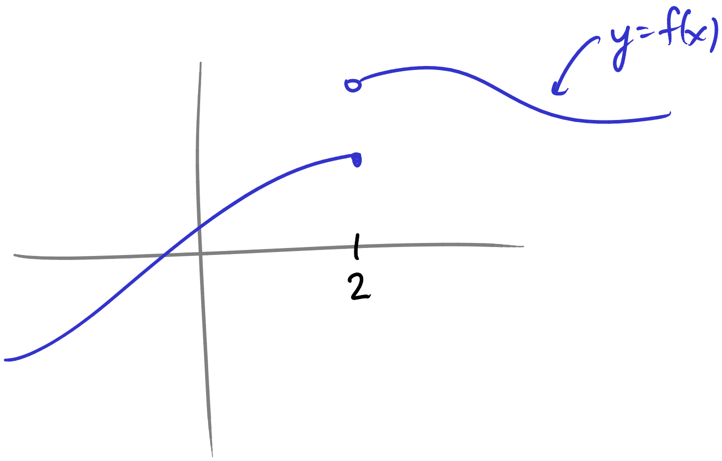
i.e. if

① $a$ is in the domain of $f$, i.e. $f(a)$ exists,

② $\lim\limits_{x \to a} f(x)$ exists,

③ $\lim\limits_{x \to a} f(x) = f(a)$.

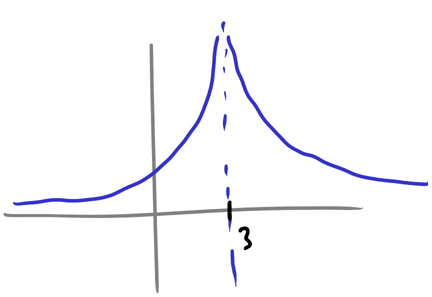
Ex




this function $f(x)$ is continuous at all $a$ except $a=2$.

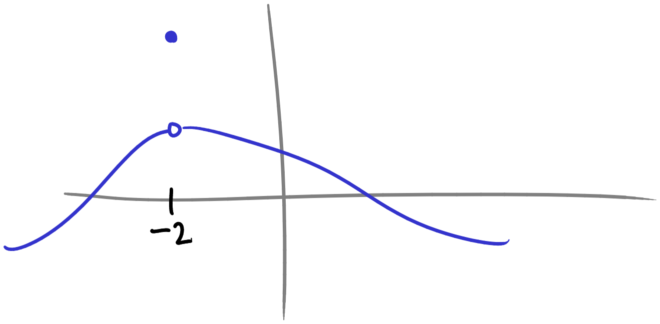
("jump discontinuity") (fails condition ②)

Ex



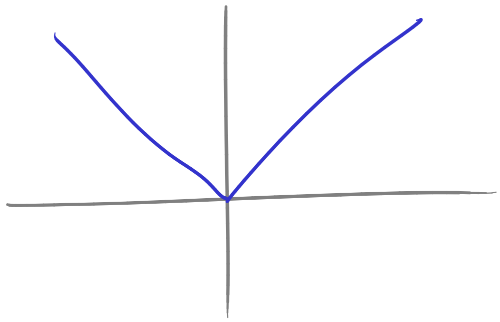

this $f(x)$ is continuous at all $a$ except $a=3$. (fails condition ①: 3 is not in domain)

Ex



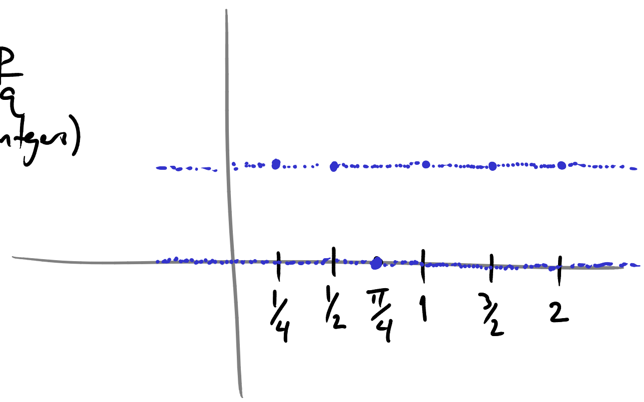

f(x) cts at all a except $a=-2$. (fails condition (3): $\lim_{x\to -2} f(x) \neq f(-2)$)

Ex



$f(x)=|x|$ cts at all a ✓

Ex $f(x)=\begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ ($x=\frac{p}{q}$ p,q integers)




f(x) is cts at no value of a ✓

Fact If f, g are cts at a then

- $f+g$ is cts at a
- $f-g$ is cts at a
- $cf$ is cts at a (c any const)
- $fg$ is cts at a
- $f/g$ is cts at a (if $g(a)\neq 0$)

Fact The following functions are continuous everywhere in their domain:

polynomials, rational functions, roots, trig functions, inverse trig functions, exp, log, composition of continuous functions

Ex $f(x)=x^2+3x+7$ is cts at all a.

$f(x)=\dfrac{x^2-3}{4}+\sin(x)$ " " " "

$f(x) = \sin\left(\frac{x+4}{x-7}\right)$ is cts at all $a$ <u>except</u> $a=7$.

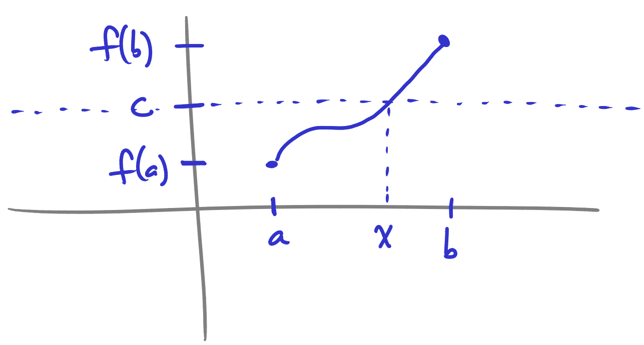
Ex $\lim_{x\to 5} \sin\left(\frac{x+4}{x-7}\right)$ is $\sin\left(\frac{5+4}{5-7}\right) = \sin\left(-\frac{9}{2}\right) = \underline{-\sin\left(\frac{9}{2}\right)}$

What about: $\lim_{x\to 0} \cos^{-1}\left(\frac{\sin x}{x}\right)$ ?

<u>Fact</u>: limits "pass through" continuous functions — if $f$ is cts at $b$, and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$.

So here, $\lim_{x\to 0} \cos^{-1}\left(\frac{\sin x}{x}\right) = \cos^{-1}\left(\lim_{x\to 0} \frac{\sin x}{x}\right) = \cos^{-1}(1) = \underline{0}$.

## Intermediate Value Theorem




If $f(x)$ is continuous at all points in the interval $[a,b]$ ("continuous on $[a,b]$")

and $f(a) < c < f(b)$

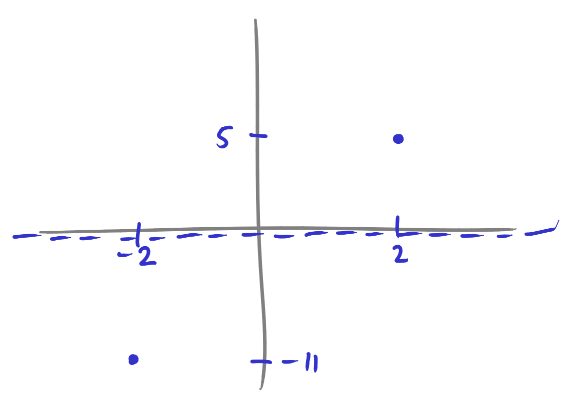
then there exists <u>some</u> $x$ in $[a,b]$ with $f(x) = c$.

Ex How do we <u>solve</u> $x^3 - x^2 + 1 = 0$ ?

Write $f(x) = x^3 - x^2 + 1$

$f(-2) = -8 - 4 + 1 = -11$

$f(2) = 8 - 4 + 1 = 5$




so IVT guarantees that

$x^3 - x^2 + 1 = 0$ has <u>at least</u> one solution in $[-2, 2]$.

**Ex** $\lim_{x \to 2} 2^{\sqrt{x^2+12}} = 2^{\sqrt{2^2+12}} = 2^{\sqrt{16}} = 2^4 = \underline{\underline{16}}$

(the function $f(x) = 2^{\sqrt{x^2+12}}$ was built from continuous functions $\Longrightarrow$ it's continuous everywhere on its domain).

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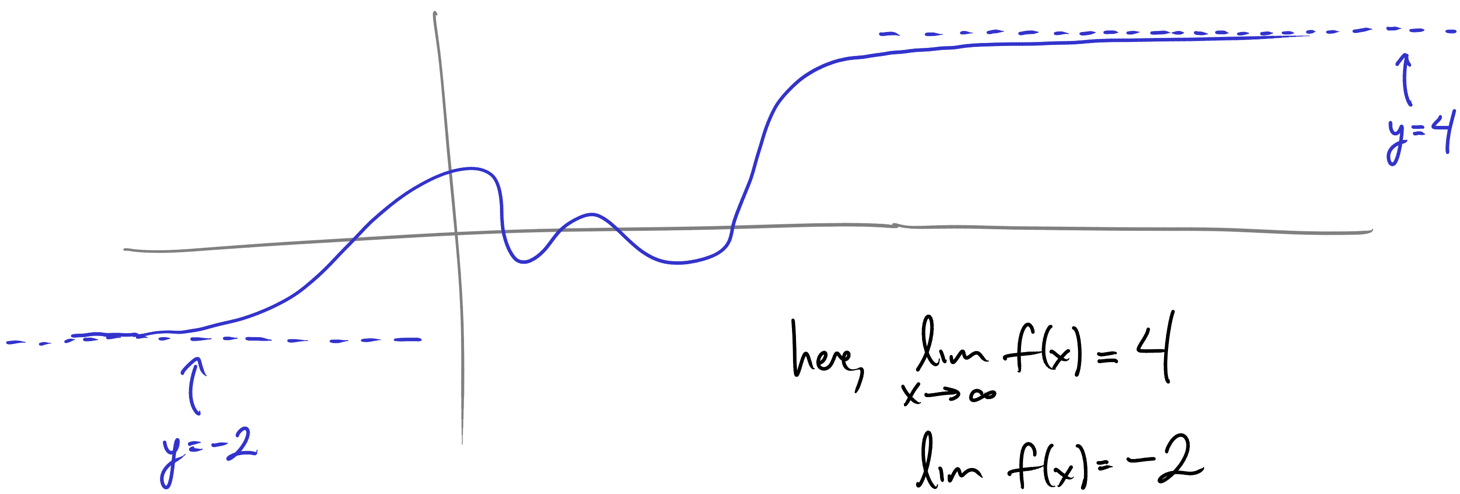
## Limits as $x \to \pm\infty$

We say $\lim_{x \to \infty} f(x) = L$ if as $x$ grows without bound in positive dir, $f(x)$ approaches $L$.

say $\lim_{x \to -\infty} f(x) = L$ if as $x$ grows without bound in negative dir, $f(x)$ approaches $L$.



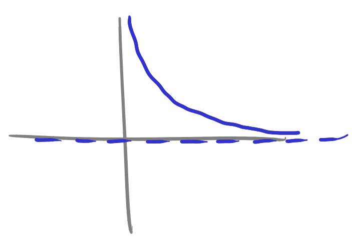

here, $\lim_{x \to \infty} f(x) = 4$

$\lim_{x \to -\infty} f(x) = -2$

If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$

we say the graph of $f(x)$ has a <u>horizontal asymptote</u> at $L$.

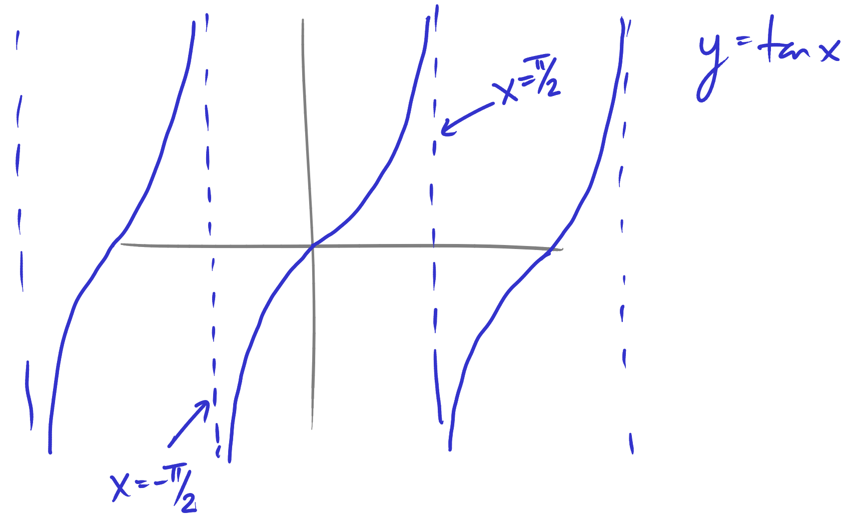
**Ex** $\lim_{x \to \infty} \frac{1}{x} = 0.$



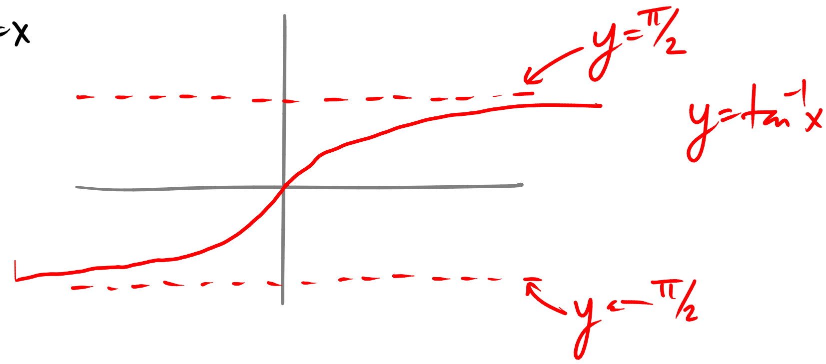
$\lim_{x \to -\infty} \frac{1}{x} = 0$

**Ex** $\lim_{x \to \infty} 6 + \frac{1}{x^2} = 6$ (plug in $x$ very large, $6 + \frac{1}{x^2} = 6 + \frac{1}{(\text{big})^2} = 6 + (\text{very small})$)

Ex $\lim_{x\to\infty} \tan^{-1} x = ?$



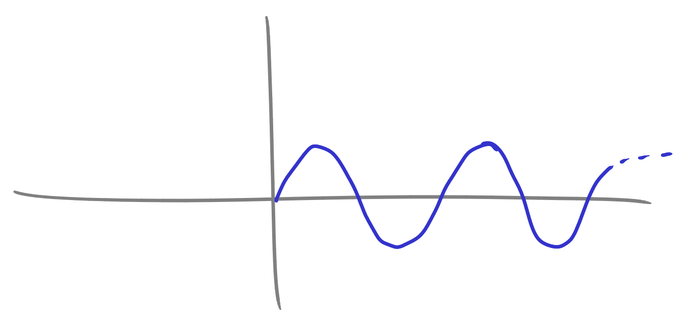

reflect the graph in the axis y=x to get graph of $\tan^{-1}x$:




So: $\lim_{x\to\infty} \tan^{-1}x = \pi/2$

$\lim_{x\to-\infty} \tan^{-1}x = -\pi/2$

Ex $\lim_{x\to\infty} \sin x$ does not exist.



(graph doesn't approach any horizontal asymptote as $x\to\infty$)

also $\lim_{x\to-\infty} \sin x$ doesn't exist.

Fact Can use Limit Laws for limits as $x\to\infty$ or $x\to-\infty$, as we did for $x\to a$.

Ex $\lim_{x\to\infty} 18 + \frac{1}{x^{4/3}} = \lim_{x\to\infty} 18 + \left(\lim_{x\to\infty}\frac{1}{x}\right)^{4/3}$

$= 18 + 0^{4/3} = 18$

Ex $\lim_{x \to \infty} \frac{x^2+4x+7}{x^2-9} = ?$

plug in very large x: get $\frac{(\text{big})}{(\text{big})}$

— not too helpful

more quantitative: if $x=1000$, get $\frac{1000^2+4000+7}{1000^2-9}$

$$= \frac{1000000+4007}{1000000-9} \approx 1$$

— looks like only the biggest power of x in num and denom matters!

ie $\lim_{x \to \infty} \frac{x^2+4x+7}{x^2-9} = \lim_{x \to \infty} \frac{x^2}{x^2} = 1.$

To prove it: $\frac{x^2+4x+7}{x^2-9} = \frac{x^2+4x+7}{x^2-9} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{1+\frac{4}{x}+\frac{7}{x^2}}{1-\frac{9}{x^2}}$

$$\lim_{x \to \infty} \frac{1+\frac{4}{x}+\frac{7}{x^2}}{1-\frac{9}{x^2}} = \frac{\lim_{x \to \infty} 1+\frac{4}{x}+\frac{7}{x^2}}{\lim_{x \to \infty} 1-\frac{9}{x^2}} = \frac{1}{1} = \underline{\underline{1}}$$

Ex $\lim_{x \to \infty} \frac{9x^3-4}{x^3+7x+8} = \underline{\underline{9}}$

Ex $\lim_{x \to \infty} \frac{x+3}{74+8x} = \underline{\underline{\frac{1}{8}}}$

Ex $\lim_{x \to \infty} \sqrt{x^2+1} - x$ plug in x big: $(\text{big})-(\text{big})$ — no help.

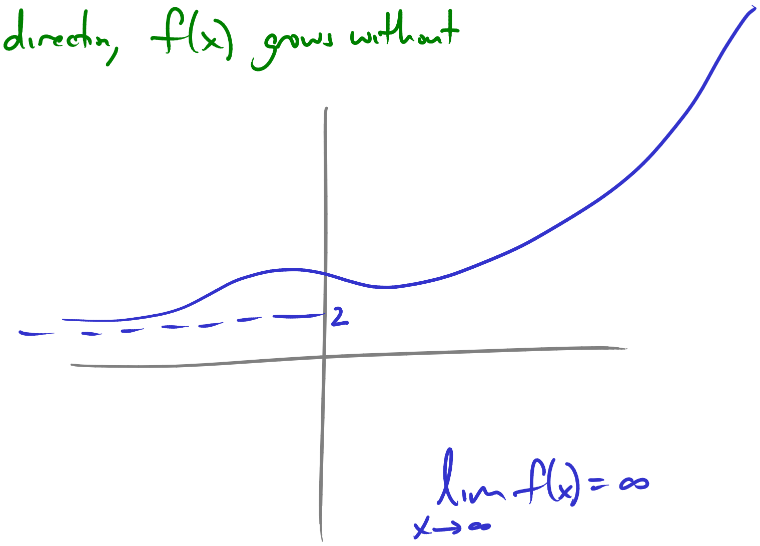
$$= \lim_{x \to \infty} \sqrt{x^2+1} - x \cdot \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$= \lim_{x \to \infty} \frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} = \lim_{x \to \infty} \frac{1}{\sqrt{x^2+1}+x} = 0$$

If, as $x$ grows without bound in the positive direction, $f(x)$ grows without bound in the positive direction, we say

$$\lim_{x \to \infty} f(x) = \infty$$

Similarly can have $\lim_{x \to \infty} f(x) = -\infty$




$$\lim_{x \to \infty} f(x) = \infty$$

$$\lim_{x \to -\infty} f(x) = 2$$

**Ex** $\lim_{x \to \infty} x = \infty$

$\lim_{x \to -\infty} x = -\infty$

**Ex** $\lim_{x \to \infty} x^2 - 10x = \infty$ ($x^2$ is <u>much bigger</u> than $10x$ when $x$ is big)

$$x^2 - 10x = \underset{\text{big positive}}{x}\,\underset{\text{big positive}}{(x-10)}$$