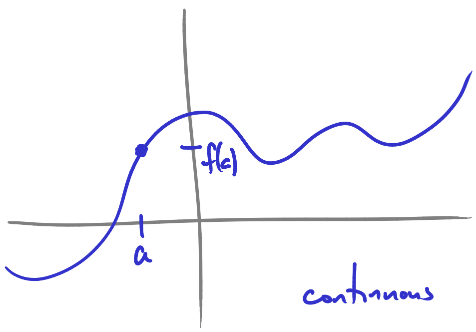
Lecture 5 — 10 Sep 2015

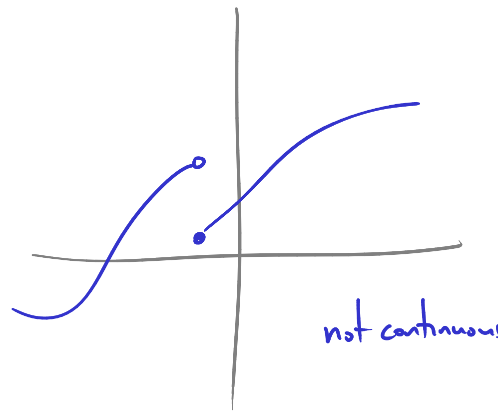
Last time: limits and their formal definition.

Continuity

Informally: we say f is continuous if "we can draw the graph of f without lifting the pencil"



continuous



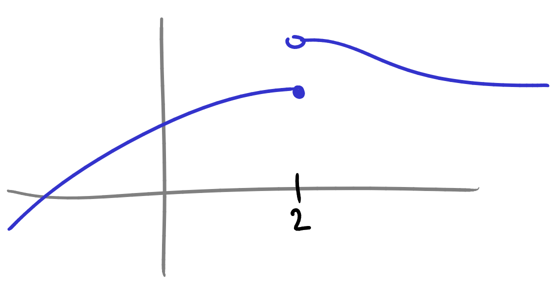
not continuous

Formally: we say f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

i.e. if
1. a is in the domain of f, so $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

Ex



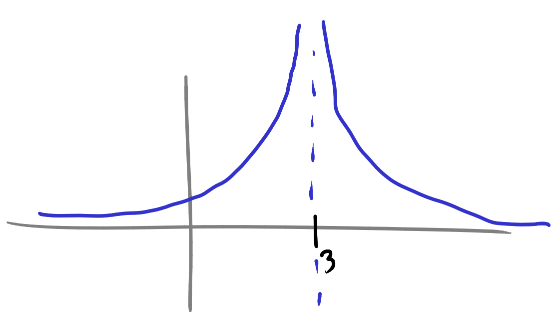
f is continuous at every a except possibly $a=2$.

At $a=2$: (1) $f(a)$ exists ✓

(2) $\lim_{x \to a} f(x)$ DNE ✗

so not cts at $a=2$.

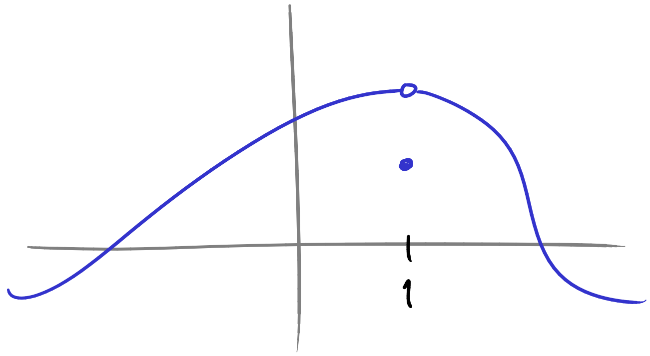
Ex



f cts at all a except maybe $a=3$.

At $a=3$: (1) a not in domain of f ✗

Ex



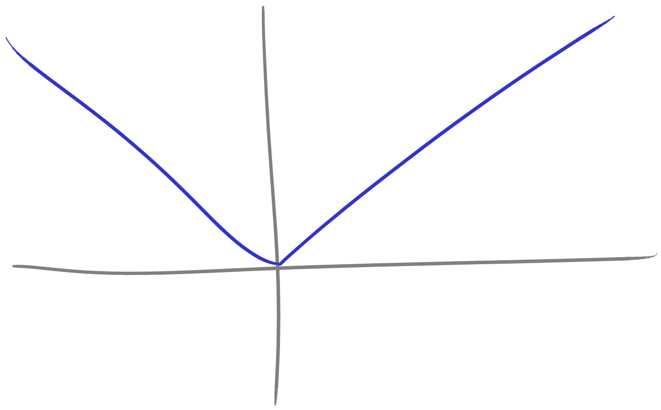
f cts at all a except a=1.

At a=1: (1),(2) ✓

(3): $\lim_{x\to a} f(x) \neq f(a)$ ✗

so <u>not</u> cts at a=1.

Ex

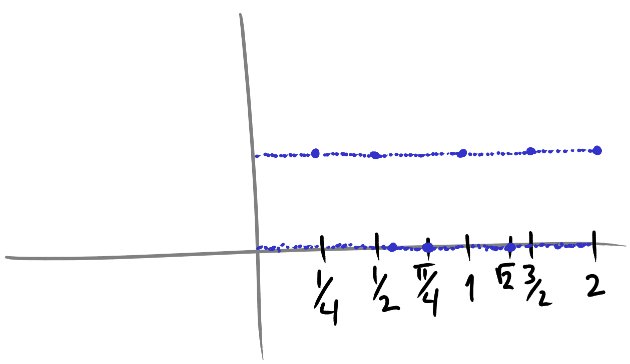


$f(x) = |x|$ check at $a=0$:

(1),(2),(3) ✓ $\lim_{x\to 0} |x| = 0 = |0|$

so <u>cts</u> at $a=0$

Ex



$$f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational} \end{cases}$$

(1) ✓

(2) ✗ not cts at <u>any</u> a.

Fact If f, g are cts at a, then

$f+g$, $f-g$, $c\cdot f$ (c any const), $f\cdot g$, f/g (if $g(a)\neq 0$)

are all cts at a.

Fact If g is cts at a, and f is cts at g(a), then $f\circ g$ is cts at a.

Fact The following functions are cts everywhere on their domains:

<u>polynomials</u>, <u>rational functions</u>, <u>roots</u>, <u>trig functions</u>, <u>inverse trig functions</u>, <u>exponentials</u>, logs

Ex $f(x) = x^2 + 3x - 1000$ is cts at all a.

$f(x) = \dfrac{x^2-3}{4\sin(x)}$ is cts at all a <u>except</u> $a = n\pi$ n integer

ie $a = 0, \pi, -\pi, 2\pi, -2\pi, \ldots$

$f(x) = \sin(1+\sqrt{x})$ is cts everywhere on its domain (ie all $a \geq 0$)

Ex $$\lim_{x\to 5} \underbrace{\sin\left(\frac{x+4}{x-7}\right)}_{\text{(continuous function of } x)} = \sin\left(\frac{5+4}{5-7}\right) = \sin\left(-\frac{9}{2}\right) = -\sin\left(\frac{9}{2}\right).$$

What about: $\displaystyle\lim_{x\to 0} \cos^{-1}\left(\frac{\sin x}{x}\right)$?

As $x \to 0$, $\frac{\sin x}{x} \to 1$, so, <u>might hope</u>: $\cos^{-1}\left(\frac{\sin x}{x}\right) \longrightarrow \cos^{-1}(1)$.

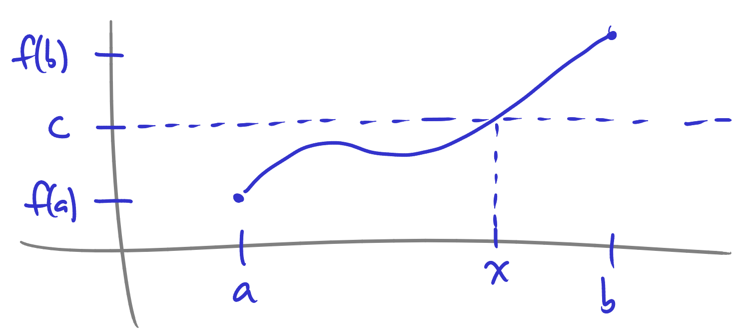
<u>This is true</u>, <u>because</u> $\cos^{-1}$ is continuous:

<u>Fact</u> If f is cts at b, and $\displaystyle\lim_{x\to a} g(x) = b$, <u>then</u> $\displaystyle\lim_{x\to a} f(g(x)) = f(b)$.

So, $$\lim_{x\to 0} \cos^{-1}\left(\frac{\sin x}{x}\right) = \cos^{-1}\left(\lim_{x\to 0} \frac{\sin x}{x}\right) = \cos^{-1}(1) = \underline{\underline{0}}.$$

<u>because</u> $\cos^{-1}$ is cts at $a=1$.

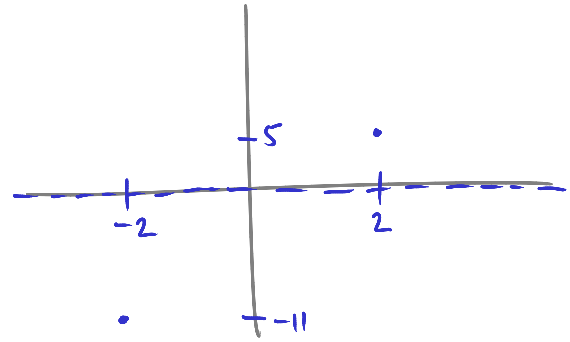
Intermediate Value Theorem



If $f(x)$ is cts at all points in the interval $[a,b]$ ("cts on $[a,b]$") and $f(a) < c < f(b)$ then there exists <u>some</u> x in $[a,b]$ such that $f(x) = c$.

Ex How do we solve $x^3 - x^2 + 1 = 0$?

Write $f(x) = x^3 - x^2 + 1$



$f(-2) = -8 - 4 + 1 = -11$

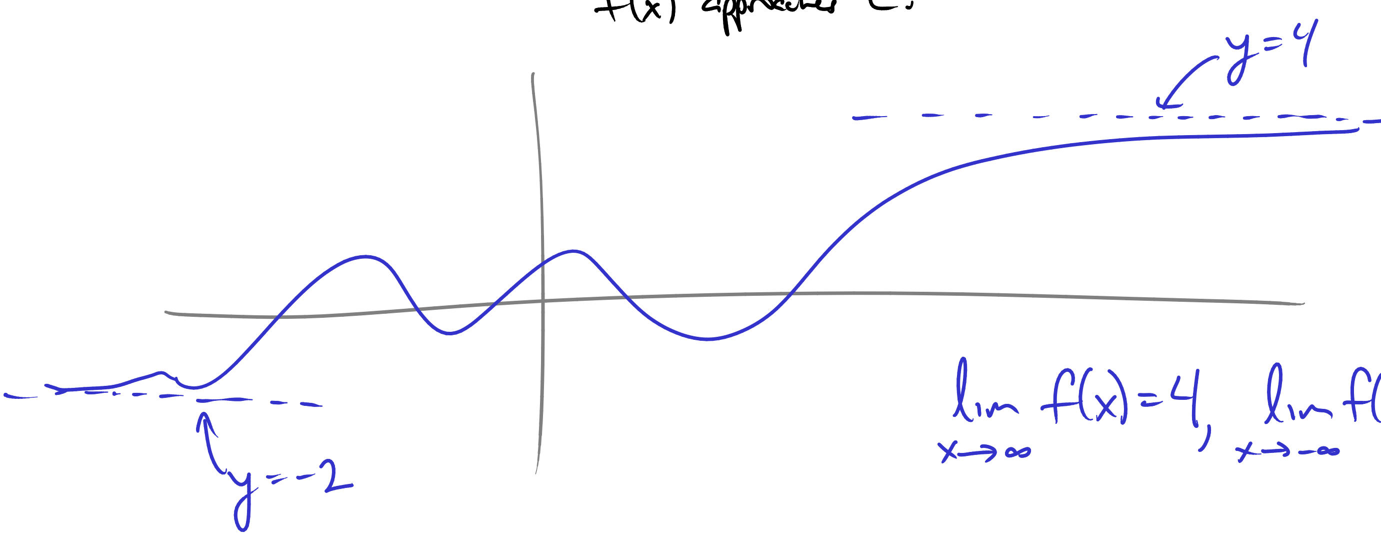
$f(2) = 8 - 4 + 1 = 5$

So by IVT, there is some x in $[-2, 2]$ such that $f(x) = 0$.

Limits as $x \to \pm\infty$

We say $\lim_{x\to\infty} f(x) = L$ if as x grows without bound in +ve direction, $f(x)$ approaches L.

Similarly: $\lim_{x\to-\infty} f(x) = L$ if as x grows without bound in −ve direction $f(x)$ approaches L.



$\lim_{x\to\infty} f(x) = 4$, $\lim_{x\to-\infty} f(x) = -2$

Ex $\lim_{x\to\infty} \frac{1}{x} = 0$ $\quad$ (why? $\frac{1}{(\text{very big})} = (\text{very small})$)

$\lim_{x\to-\infty} \frac{1}{x} = 0$

Ex $\lim_{x\to\infty} \sin x$ does not exist (doesn't approach any horizontal asymptote)

Ex $\lim_{x\to\infty} 18+\frac{1}{x^2} = \lim_{x\to\infty} 18 + \lim_{x\to\infty}\frac{1}{x^2}$ ← (can use Limit Laws for $\lim_{x\to\infty}$ just like for $\lim_{x\to a}$)

$= 18 + \left(\lim_{x\to\infty}\frac{1}{x}\right)^2$

$= 18 + 0^2 = 18$

Ex $\lim_{x\to\infty} \frac{x^2+4x+19}{x^2-8x-1} = \lim_{x\to\infty} \frac{x^2+4x+19}{x^2-8x-1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \lim_{x\to\infty}\frac{1+\frac{4}{x}+\frac{19}{x^2}}{1-\frac{8}{x}-\frac{1}{x^2}} = \frac{\lim_{x\to\infty} 1+\frac{4}{x}+\frac{19}{x^2}}{\lim_{x\to\infty} 1-\frac{8}{x}-\frac{1}{x^2}}$

$= \frac{1}{1} = \underline{\underline{1}}$

Ex $\lim_{x\to\infty}\frac{3x^3+4}{10x^3-7x} = \underline{\underline{\frac{3}{10}}}$

If as $x\to\infty$, $f(x)$ increases without bound, then we say $\lim_{x\to\infty} f(x) = \infty$. Similarly for $-\infty$.

Ex $\lim_{x\to\infty}\frac{7x^2+8x}{x-2} = \infty$ (why? $\frac{7x^2+8x}{x-2} = x\cdot\frac{7x+8}{x-2}$

$= x\cdot\frac{7+\frac{8}{x}}{1-\frac{2}{x}}$

→ about $7\cdot x$ if x is very big

so → ∞ as $x\to\infty$)

Ex $\lim_{x\to\infty}\frac{x+4}{x^3-7} = 0$ (why? $\approx \frac{x}{x^3} = \frac{1}{x^2}$ which → 0 as $x\to\infty$)

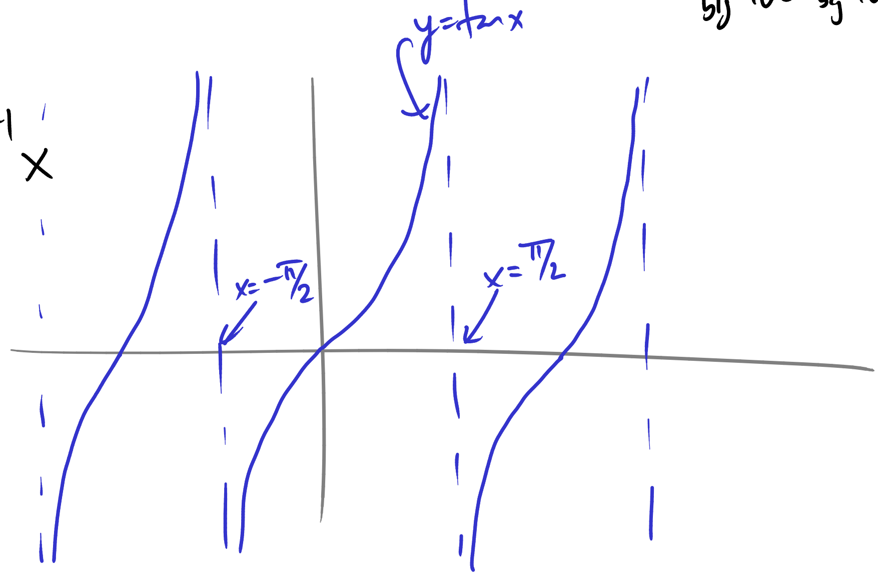
Ex $\lim\limits_{x\to\infty} \dfrac{x^2+3}{1-x} = -\infty$ (why? $\approx \dfrac{x^2}{-x} = -x$ which $\to -\infty$ as $x\to\infty$)

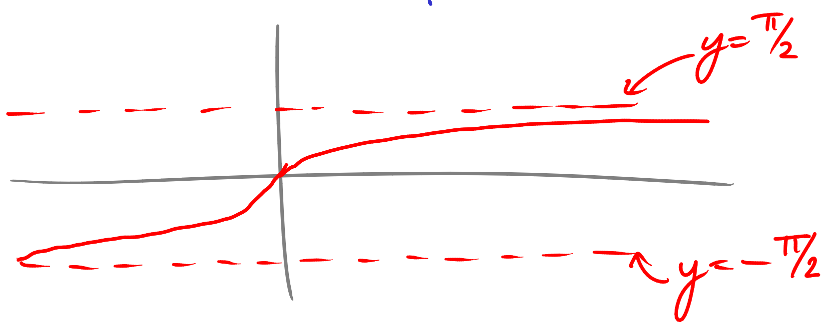
Ex $\lim\limits_{x\to\infty} x^2-10x = \infty$ (why? $\approx x^2$

or, $x^2-10x = \underset{\text{big +ve}}{x}\,\underset{\text{big +ve}}{(x-10)}$)

Ex $\lim\limits_{x\to\infty} \tan^{-1} x = \dfrac{\pi}{2}$

$\lim\limits_{x\to-\infty} \tan^{-1} x = -\dfrac{\pi}{2}$





Ex $\lim\limits_{x\to\infty} \sqrt{x^2+1} - x = ?$

$$\sqrt{x^2+1} - x \cdot \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} = \frac{1}{\sqrt{x^2+1}+x}$$

if x very big, this is $\dfrac{1}{(\text{very big})}$ so get $\lim\limits_{x\to\infty} \sqrt{x^2+1} - x = 0$.

Ex $\lim\limits_{x\to\infty} \sqrt{x+1} - \sqrt{x} = ?$

$$\left(\sqrt{x+1}-\sqrt{x}\right)\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} = \frac{(x+1)-(x)}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{\sqrt{x+1}+\sqrt{x}}$$

$\lim = 0$ ✓

Fact If $f(x) = \dfrac{P(x)}{Q(x)}$ rational function, then:

- if degree(P) = degree(Q), then $\lim_{x\to\infty} f(x)$ = ratio of leading coefficients
  $\lim_{x\to-\infty} f(x)$ = " " " "

  e.g. $\displaystyle\lim_{x\to-\infty} \frac{4x^3-1}{8x^3+8x} = \frac{4}{8} = \frac{1}{2}$

- if degree(P) < degree(Q), $\lim_{x\to\infty} f(x) = 0$
  $\lim_{x\to-\infty} f(x) = 0$

- if degree(P) > degree(Q), $\lim_{x\to\infty} f(x) = \pm\infty$
  $\lim_{x\to-\infty} f(x) = \pm\infty$
  Sign determined by sign of leading coefficients.

  e.g. $\displaystyle\lim_{x\to\infty} \frac{x^2+1}{-x} = \underline{\underline{-\infty}}$

Ex $\displaystyle\lim_{x\to\infty} x\left(\sqrt{1+\frac{1}{x}}-1\right)$

$$= \lim_{x\to\infty} x\left(\sqrt{1+\tfrac{1}{x}}-1\right)\frac{\sqrt{1+\frac{1}{x}}+1}{\sqrt{1+\frac{1}{x}}+1} = \lim_{x\to\infty} x\,\frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x}}+1}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} = \underline{\underline{\frac{1}{2}}}$$