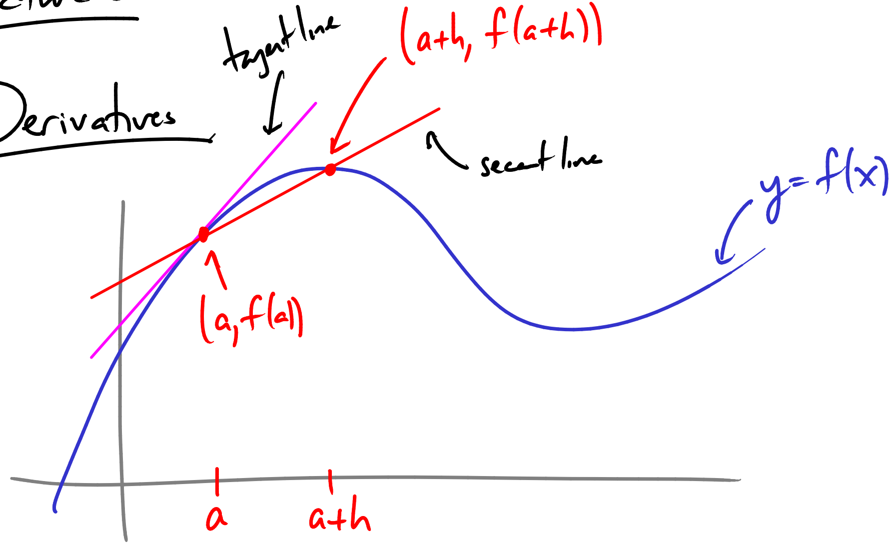
# Lecture 6

15 Sep 2015

## Derivatives



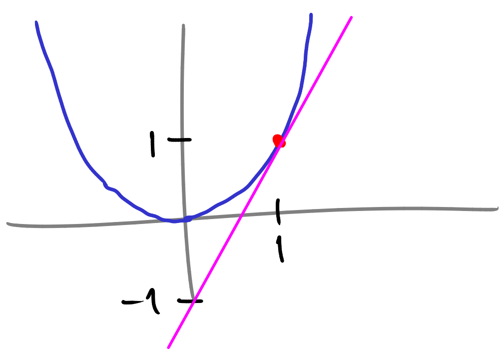

slope of secant line $= \dfrac{\text{rise}}{\text{run}}$

$$= \frac{f(a+h) - f(a)}{(a+h) - a}$$

$$= \frac{f(a+h) - f(a)}{h}$$

slope of tangent line at $(a, f(a))$ $= \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ (if this limit exists!)

**Ex** What is the tangent line to the graph $y = x^2$ at the point $(1,1)$? $\qquad f(x) = x^2$




$$\text{slope} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} 2 + h = \underline{\underline{2}}$$

Line thru $(1,1)$ with slope 2:

$$y - 1 = 2(x-1) \quad \text{ie} \quad y - 1 = 2x - 2$$

$$\underline{\underline{y = 2x - 1}}$$

We call the slope "derivative":

The derivative of a function $f$ at a number $a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Ex ① If $f(x) = 7x^2 - 3x + 1$ what is $f'(x)$?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left(7(x+h)^2 - 3(x+h) + 1\right) - \left(7x^2 - 3x + 1\right)}{h}$$

$$= \lim_{h\to 0} (\cdots) = \lim_{h\to 0} \frac{14xh - 3h + 7h^2}{h} = \lim_{h\to 0} 14x - 3 + 7h$$

$$= \underline{14x - 3}$$

② What is the tangent line to the graph of $y = f(x)$ at $(-1, 11)$?

Slope is $f'(-1) = 14(-1) - 3 = -17$

Line w/slope $-17$ thru $(-1,11)$ is

$$y - 11 = -17(x - (-1))$$

$$y - 11 = -17x - 17$$

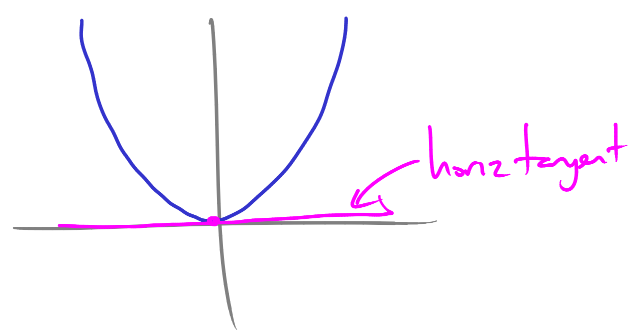
$$\underline{y = -17x - 6}$$

Ex Where does the graph of $y = x^2$ have a horizontal tangent line?

i.e. if $f(x) = x^2$, where does $f'(x) = 0$? (horizontal $\Leftrightarrow$ slope $= 0$)

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \cdots = 2x$$

So $f'(x) = 0$ only at $x = 0$.
$(2x = 0)$

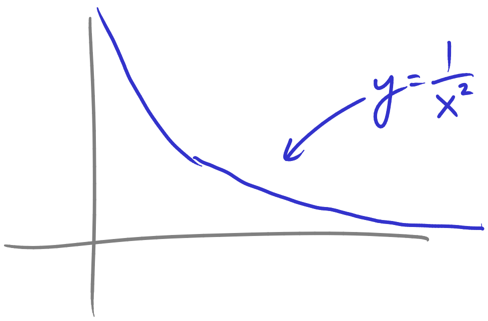



Ex If $f(x) = \frac{1}{x^2}$ what is $f'(x)$?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

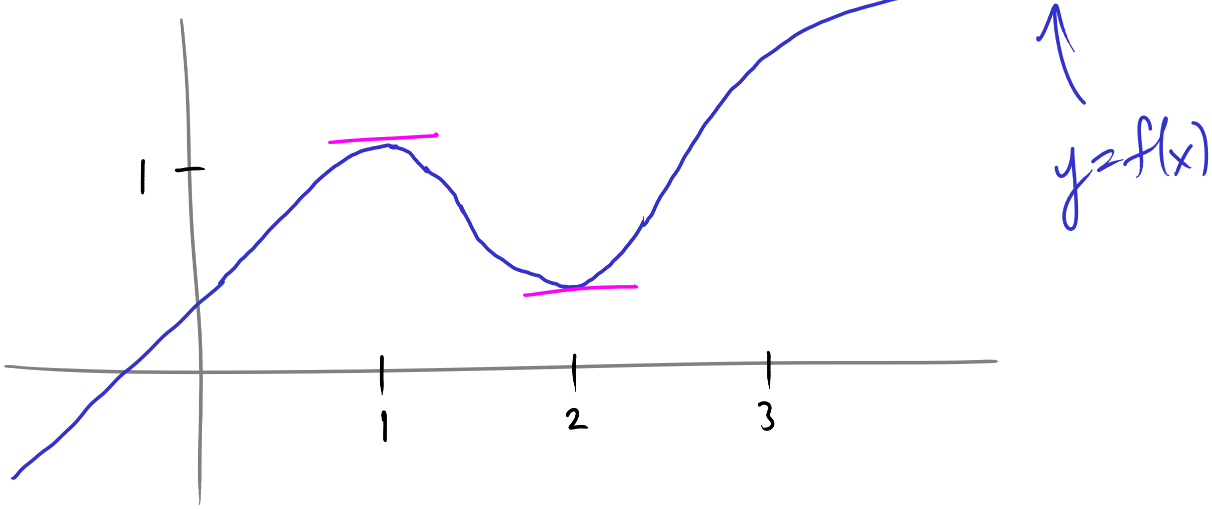
$$= \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{(x+h)^2x^2}}{h} = \lim_{h\to 0} \frac{x^2-(x+h)^2}{h\cdot(x+h)^2x^2} = \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{h\cdot(x+h)^2x^2}$$

$$= \lim_{h\to 0} -\frac{2xh+h^2}{h\cdot(x+h)^2\cdot x^2} = \lim_{h\to 0} -\frac{2x+h}{(x+h)^2x^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}$$

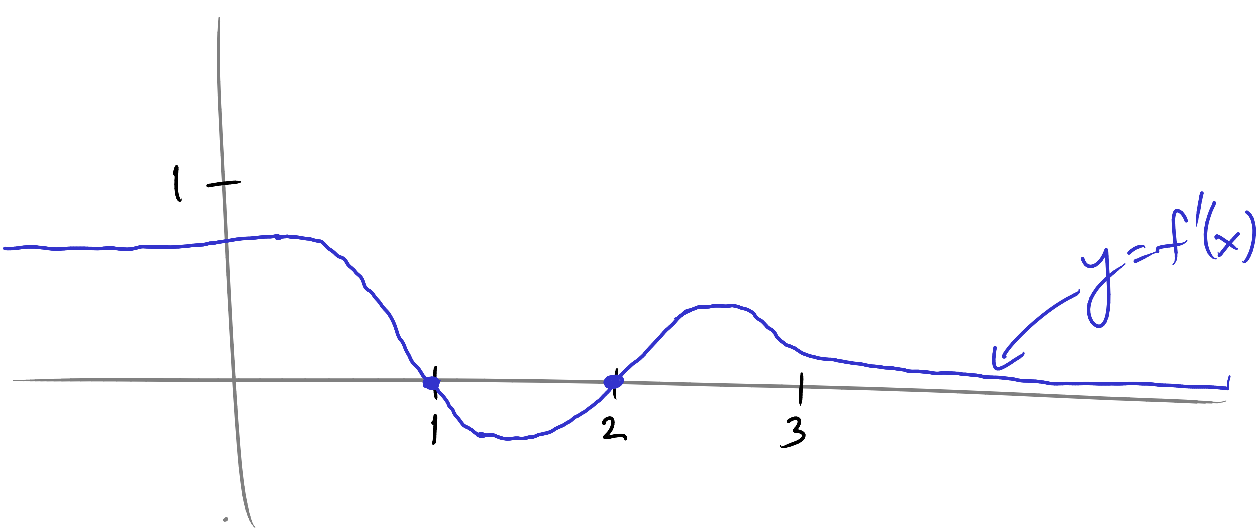



Check: for $x>0$ slope of tangent line is negative, so <u>should</u> have $f'(x)<0$ and indeed, $-\frac{2}{x^3}<0$ ✓

<u>Ex</u> If $f(x)$ is

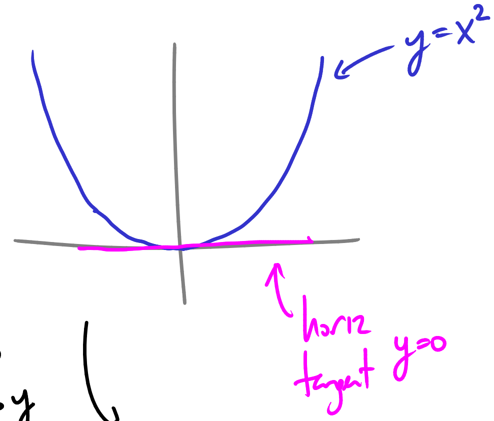



sketch $f'(x)$.

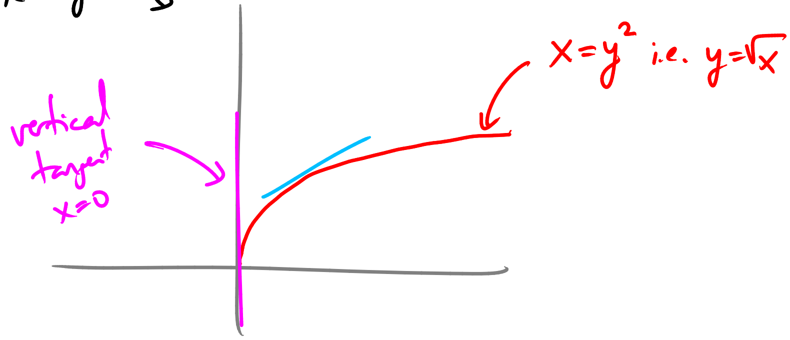


Ex If $f(x)=\sqrt{x}$

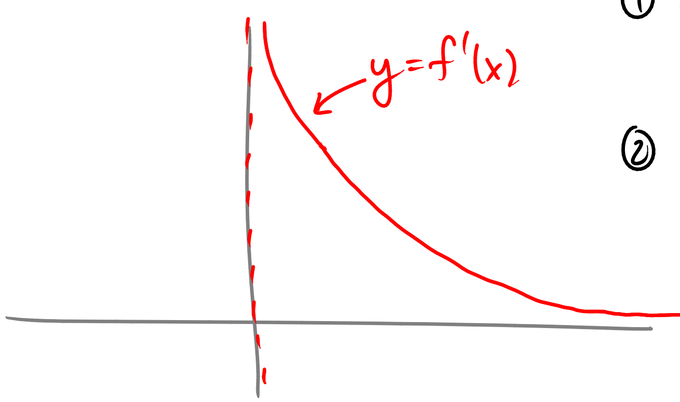
① sketch $f'(x)$




swap $x \leftrightarrow y$




$f(x)=\sqrt{x}$




① slope of tangent line always positive $\rightarrow f'(x)>0$

② as $x\to 0$, tangent line gets closer to being vertical

$\rightarrow \lim_{x\to 0^+} f'(x) = +\infty$

② Calculate $f'(x)$. $\quad f(x)=\sqrt{x}$

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \lim_{h\to 0}\frac{(x+h)-x}{h\cdot(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{h}{h\cdot(\sqrt{x+h}+\sqrt{x})}$$

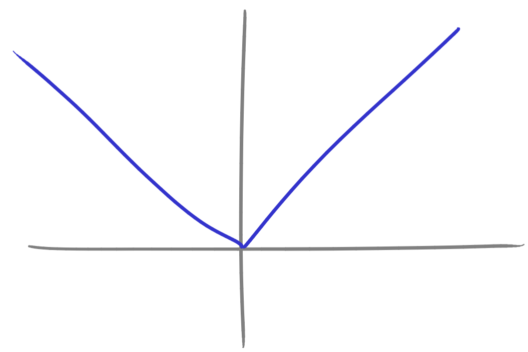
$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

We say $f(x)$ is <u>differentiable</u> at $x$ if $f'(x)$ exists, is <u>not</u> $= \pm\infty$.

(i.e. $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ exists, not $\pm\infty$.)

<u>Ex</u> $f(x)=x^2$ is differentiable at all real #'s $x$. $\left(\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = 2x\right)$

<u>Ex</u> Where is $f(x)=|x|$ differentiable?



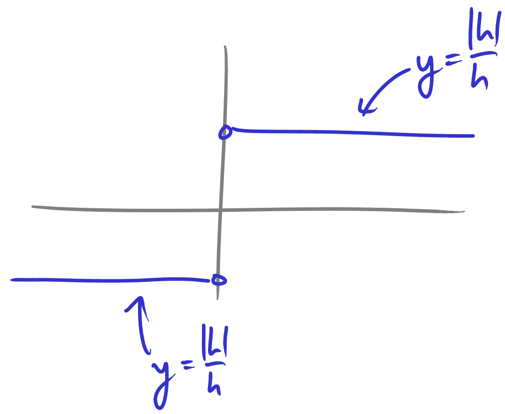
For $x>0$, $f'(x)=1$

$x<0$, $f'(x)=-1$

$x=0$, $f'(x)=\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h}$

$= \lim\limits_{h\to 0} \dfrac{|h|-|0|}{h} = \lim\limits_{h\to 0} \dfrac{|h|}{h}$

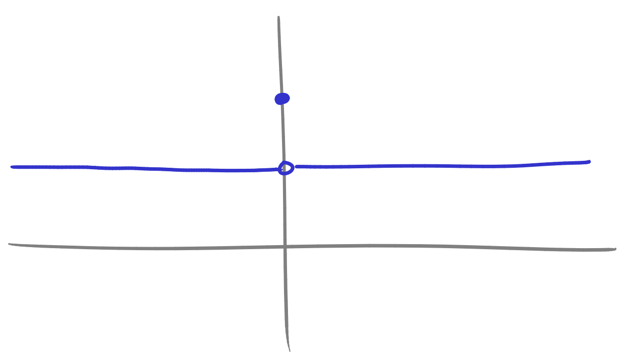
<u>DNE</u>




so $f(x)$ is differentiable at all $x$ <u>except</u> $x=0$.

(Generally: <u>sharp point</u> $\longrightarrow$ <u>not differentiable</u>)

<u>Ex</u> $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x=0 \end{cases}$



For $x\neq 0$, $f'(x)=0$

for $x=0$, $f'(x) = \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h}$

for $h$ small: $\dfrac{f(h)-f(0)}{h} = \dfrac{1-2}{h} = -\dfrac{1}{h}$

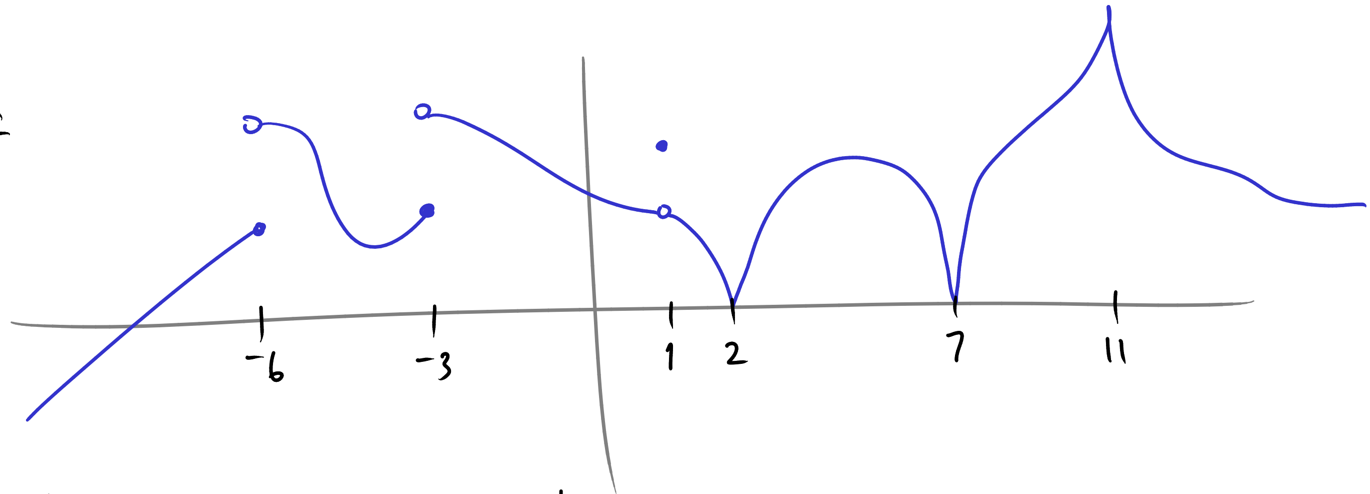
($h\neq 0$)

so $\lim\limits_{h\to 0}$ DNE.

So $f(x)$ not differentiable at $x=0$.

General rule: if $f$ not continuous then it's not differentiable.

Ex




differentiable at all $x$ except at $x = -6, -3, 1, 2, 7, 11$

Interpretation of $f'(x)$

If $x(t)$ is the position of an object at time $t$ seconds (in meters)

then $x'(t)$ is the velocity of the object at time $t$. (in meters/sec)

$$\overset{\displaystyle x'(t)}{\underset{\displaystyle v(t)}{\parallel}}$$

Ex An electron in a uniform electric field moves as

$$x(t) = \frac{1}{2}t^2$$

What is its velocity at time $t$?

$$v(t) = x'(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = \lim_{h \to 0} \frac{\frac{1}{2}(t+h)^2 - \frac{1}{2}t^2}{h}$$

$$= \cdots = \underline{\underline{t}}$$

Is it speeding up or slowing down? speeding up

In general, if $t$ = time

then $f'(t)$ is the rate of change of $f(t)$.

Ex if $V(t)$ is the volume of water in Lake Travis (in gal) at time $t$ (in sec)

then $V'(t)$ is the rate water is entering/leaving the lake (in $\frac{gal}{sec}$)