Lecture 7

17 Sep 2015

- Exam 1 Sep 29 (week from Tue)
  - In class
  - only need pencils, erasers
  - covers everything in HW up to the exam date
  - questions drawn from Quest bank (≈ 20)
  - cannot use calculators

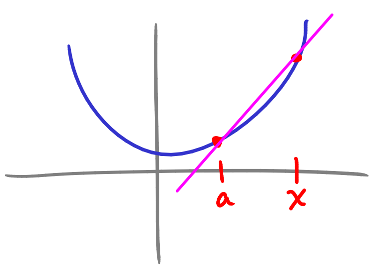
- "Problems Plus" pp. 170–171 #3, 4, 5, 8 for fun!

Last time: derivatives

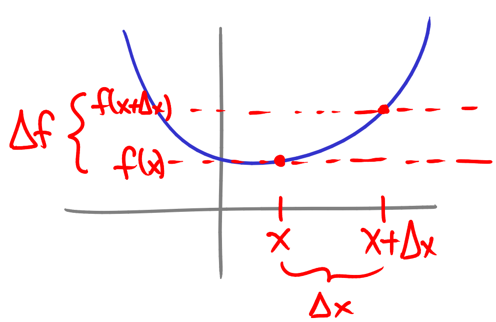
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

sometimes more convenient to write this as:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



or: $$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$



We can also repeat:

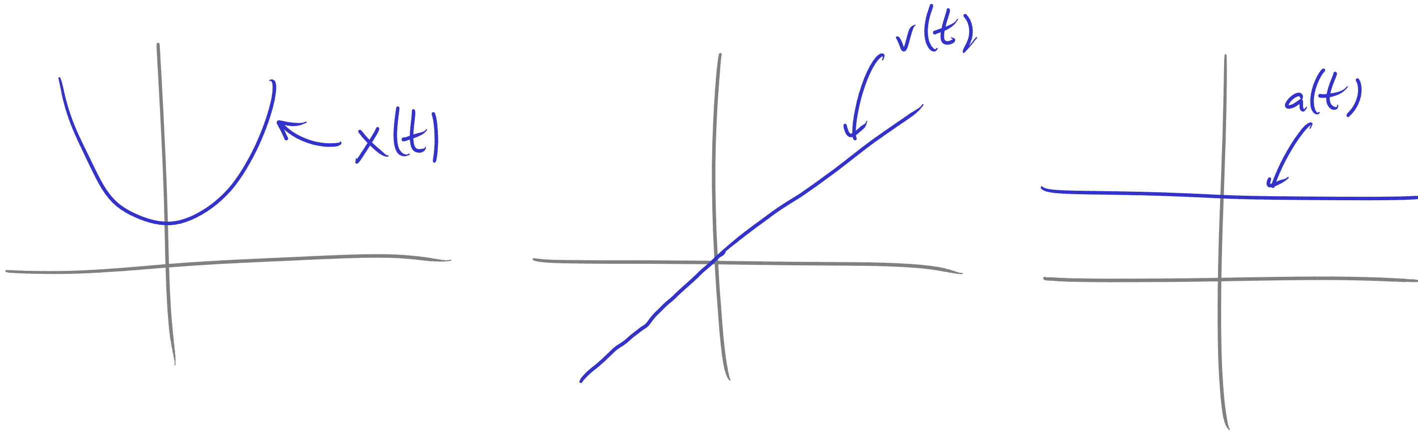
$$f''(x) = \text{second derivative of } f(x) = \text{derivative of } f'(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

$f''(x) =$ "rate of change of the rate of change of $f(x)$"

If $x(t)$ is <u>position</u> of something at time t

$v(t) = x'(t)$ is <u>velocity</u> at time t

$a(t) = v'(t) = x''(t)$ is <u>acceleration</u> at time t



Another notation:

$\dfrac{df}{dx}$ or $\dfrac{d}{dx}f(x)$ means $f'(x)$

$\dfrac{d^2f}{dx^2}$ or $\dfrac{d^2}{dx^2}f(x)$ means $f''(x)$

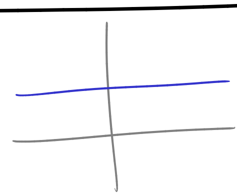
$\vdots$

$\dfrac{d^nf}{dx^n}$ or $\dfrac{d^n}{dx^n}f(x)$ or $f^{(n)}(x)$ means the n^{th} derivative of $f(x)$

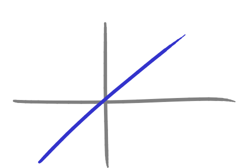
e.g. $f^{(57)}(x) = 57^{th}$ deriv. of $f(x)$

<u>Ex</u> $\dfrac{d}{dx}(x^2) = 2x$ $\quad \dfrac{d}{dx}(2x) = 2$ $\quad$ so $\dfrac{d^2}{dx^2}(x^2) = 2$

Recall: $\dfrac{d}{dx}(c) = 0$



$\dfrac{d}{dx}(x) = 1$



Fact: $\frac{d}{dx}(x^n) = nx^{n-1}$ for any integer n

Why? Say $f(x) = x^n$ $\quad f'(a) = \lim\limits_{x \to a} \frac{x^n - a^n}{x-a}$

and $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + x^{n-4}a^3 + \cdots + xa^{n-2} + a^{n-1})$

e.g. $x^4 - a^4 = (x-a)(x^3 + x^2a + xa^2 + a^3)$

So, $\lim\limits_{x \to a} \frac{x^n - a^n}{x-a} = \lim\limits_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + a^{n-1})}{x-a}$

$= \lim\limits_{x \to a} x^{n-1} + x^{n-2}a + x^{n-3}a^2 + x^{n-4}a^3 + \cdots + xa^{n-2} + a^{n-1}$

$= a^{n-1} + a^{n-2} \cdot a + a^{n-3} \cdot a^2 + a^{n-4} \cdot a^3 + \cdots + a^{n-1}$

$= \underbrace{a^{n-1} + a^{n-1} + a^{n-1} + a^{n-1} + \cdots + a^{n-1}}_{n \text{ terms}}$

$= n \cdot a^{n-1}$

Ex $\frac{d}{dx}(x^4) = 4x^{4-1} = 4x^3$

$\frac{d}{dt}(t^9) = 9t^{9-1} = 9t^8$

Actually this works for any power:

Power rule $\frac{d}{dx}(x^r) = rx^{r-1}$ for any constant r

Ex $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

Ex $\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}(x^{-3}) = -3x^{-3-1} = -3x^{-4} = \frac{-3}{x^4}$

Constant Multiple Rule $\frac{d}{dx}(c\cdot f(x)) = c\cdot \frac{d}{dx} f(x)$ c any constant, f(x) differentiable at x

$$\left[\text{Why? } \lim_{h\to 0} \frac{c\cdot f(x+h) - c\cdot f(x)}{h} = c\cdot \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right]$$

Ex $\frac{d}{dx}\left(\frac{-7}{x^2}\right) = -7\frac{d}{dx}\left(\frac{1}{x^2}\right) = -7\frac{d}{dx}(x^{-2}) = -7\left(-2x^{-3}\right) = 14x^{-3} = \frac{14}{x^3}$

Ex $\frac{d}{dx}\left(0\cdot \frac{\sin(x^{17}+\sqrt{17})}{\tan(\sqrt{x^3-1})}\right) = 0$

Ex $\frac{d}{dr}(\pi r^2) = \pi\frac{d}{dr}(r^2) = \pi\cdot 2r = 2\pi r$ (why?)

πr^2: area of a circle; $2\pi r$: circumference of a circle

Sum Rule $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ if both f, g are differentiable at x

(exercise: prove this using definition of $\frac{d}{dx}$)

Ex $\frac{d}{dx}\left(x+\frac{1}{x}\right) = \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{1}{x}\right)$

$= 1 - \frac{1}{x^2}$ (power rule)

Ex $\frac{d}{dx}(x^2-3x) = 2x-3$

Ex $\frac{d}{dx}\left(\frac{3-5x^3}{\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{3}{\sqrt{x}}\right) + \frac{d}{dx}\left(\frac{-5x^3}{\sqrt{x}}\right)$ $\quad \frac{x^3}{x^{1/2}} = x^{3-1/2} = x^{5/2}$

$= \frac{d}{dx}(3x^{-1/2}) + \frac{d}{dx}(-5x^{5/2})$

$= 3\cdot\left(-\frac{1}{2}x^{-3/2}\right) + -5\cdot\left(\frac{5}{2}x^{3/2}\right)$

$= -\frac{3}{2}x^{-3/2} - \frac{25}{2}x^{3/2}$

Exponential functions

Fact There is a number, e, such that $\displaystyle\lim_{h\to 0} \frac{e^h-1}{h} = 1.$

$(e \approx 2.71828\ldots)$

Fact $\displaystyle\frac{d}{dx}(e^x) = e^x$ (NOT xe^{x-1})

$$\left[\text{Why? } \frac{d}{dx}(e^x) = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = \lim_{h\to 0} e^x \cdot \frac{e^h-1}{h} = e^x \lim_{h\to 0}\frac{e^h-1}{h} = e^x.\right]$$

Ex $\displaystyle\frac{d}{dx}(x+13e^x) = 1+13e^x$

Product Rule $\displaystyle\frac{d}{dx}\big(f(x)\cdot g(x)\big) = f'(x)g(x) + f(x)g'(x)$ (NOT $f'(x)g'(x)$)

Ex $$\frac{d}{dx}\Big(\underset{f}{\underset{\uparrow}{x^3}}\,\underset{g}{\underset{\uparrow}{e^x}}\Big) = \underset{f'}{\underset{\uparrow}{3x^2}}\,\underset{g}{\underset{\uparrow}{e^x}} + \underset{f}{\underset{\uparrow}{x^3}}\,\underset{g'}{\underset{\uparrow}{e^x}} = \underline{\underline{(3x^2+x^3)e^x}}$$

Ex $$\frac{d}{dt}\Big(\underset{f}{\underset{\uparrow}{\sqrt{t}}}\,\underset{g}{\underset{\uparrow}{(a+bt)}}\Big) = \underset{f'}{\underset{\uparrow}{\frac{1}{2\sqrt{t}}}}\,\underset{g}{\underset{\uparrow}{(a+bt)}} + \underset{f}{\underset{\uparrow}{\sqrt{t}}}\cdot\underset{g'}{\underset{\uparrow}{b}}$$

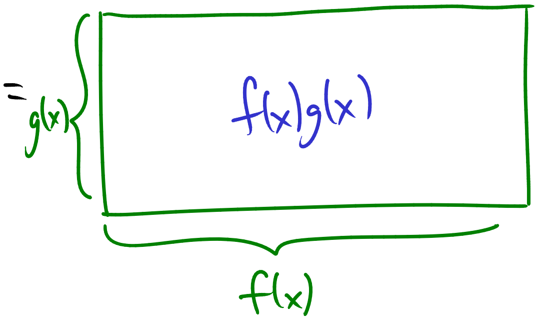
$$= \frac{a}{2\sqrt{t}} + \frac{bt}{2\sqrt{t}} + b\sqrt{t}$$

$$= \frac{a}{2\sqrt{t}} + \frac{b}{2}\sqrt{t} + b\sqrt{t} = \underline{\underline{\frac{a}{2\sqrt{t}} + \frac{3b}{2}\sqrt{t}}}$$

(note: can also get same answer by multiplying out, using Power Rule)

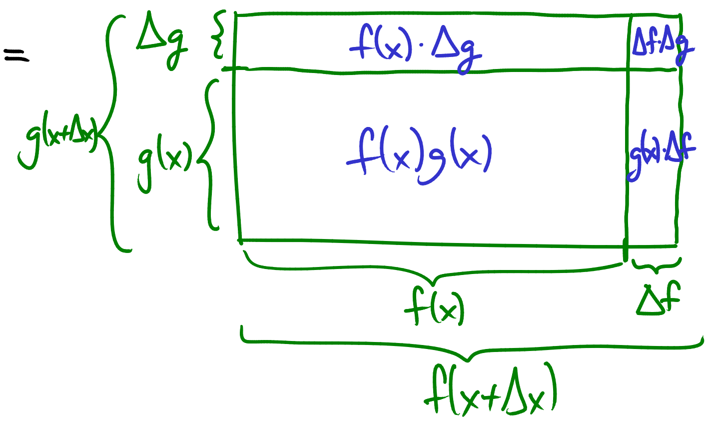
Why does Product Rule work?

$f(x)g(x) =$ area of rectangle



change x a little bit, to $x+\Delta x$:

$f(x+\Delta x)g(x+\Delta x) =$



$$\Delta(fg) = f(x)\Delta g + g(x)\Delta f + \Delta f \Delta g$$

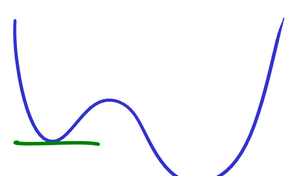
$$\frac{d}{dx}f(x)g(x) = \lim_{\Delta x\to 0}\frac{\Delta(fg)}{\Delta x} = \lim_{\Delta x\to 0} f(x)\frac{\Delta g}{\Delta x} + g(x)\frac{\Delta f}{\Delta x} + \frac{\Delta f}{\Delta x}\frac{\Delta g}{\Delta x}\Delta x$$

$$= f(x)g'(x) + g(x)f'(x) + \cancel{f'(x)g'(x)\lim_{\Delta x\to 0}\Delta x}^{\,0}$$

<u>Ex</u> $\frac{d}{dx}(e^{2x}) = \frac{d}{dx}(\underset{f}{e^x}\cdot\underset{g}{e^x}) = \underset{f'}{e^x}\cdot\underset{g}{e^x} + \underset{f}{e^x}\cdot\underset{g'}{e^x} = e^{2x}+e^{2x} = \underline{2e^{2x}}$

<u>Ex</u> Where does the graph $y=\frac{e^x}{x}$ have a horiz. tangent?

This means $\frac{dy}{dx}=0$.



$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x\cdot\frac{1}{x}\right) = e^x\cdot\frac{1}{x} + e^x\cdot\left(-\frac{1}{x^2}\right)$$

$$= e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$$

never 0

$= 0$ if $\frac{1}{x} = \frac{1}{x^2}$ i.e. $x = 1$