Lecture 11

6 Oct 2015

Exam result: ~ 87%

Office hr today (T) 4–5:30

Next HW due Fri night (Sat morning) 3am

Last time: implicit differentiation

derivatives of logarithmic and inverse trig functions

Exponential growth and exponential decay

Suppose we have a function $P(t)$ such that $\frac{dP}{dt}$ is proportional to P itself

ie $\quad \frac{dP}{dt} = kP \quad$ for some constant k.

Then what is $P(t)$?

One possibility: $P(t) = e^{kt} \qquad \frac{dP}{dt} = ke^{kt} = kP$

$P(t) = 0 \qquad \frac{dP}{dt} = 0 = k \cdot 0 = k \cdot P$

$$P(t) = \left(\frac{a^t}{\ln a}\right) \quad \frac{d}{dt} P(t) = a^t \cdot \frac{\ln a}{\ln a} = a^t = (\ln a) \cdot \frac{a^t}{\ln a}$$

$$= (\ln a) \cdot P$$

$\checkmark$ if $k = \ln a$

Most general possibility: $P(t) = Ce^{kt}$

$$\left[\text{then } \frac{dP}{dt} = C \cdot k e^{kt} = k \cdot Ce^{kt} = kP(t)\right]$$

So: $\frac{dP}{dt} = kP(t) \Longrightarrow P(t) = Ce^{kt}$ for some C.

Examples: (1) population growth under constant conditions $(k>0)$

(2) radioactive decay $(k<0)$

(3) compound interest

Ex A population of bacteria grows from 1 g at 2pm to 15 g at 5pm.

What will be the mass of bacteria at 10 pm?

$$P(t) = Ce^{kt}$$

C, k unknown

$t =$ # hours past 2pm

P measured in grams

$$P(0) = 1 \Rightarrow Ce^{k0} = 1$$

ie $C = 1$

$$P(3) = 15 \Rightarrow Ce^{k\cdot 3} = 15$$

$$e^{k\cdot 3} = 15$$

$$3k = \ln 15$$

$$k = (\ln 15)/3$$

We want $P(8)$. $P(8) = Ce^{k\cdot 8} = 1\cdot e^{\left(\frac{\ln 15}{3}\right)\cdot 8} = \underline{\underline{e^{\frac{8}{3}\ln(15)}}}$

≈ 1368 g bacteria

Ex The $\frac{1}{2}$-life of radium-226 is 1590 yrs.

Suppose we have 100mg of radium-226.

When will it be reduced to 30mg?

P in mg

$$P(t) = Ce^{kt} \qquad P(0) = 100 \Rightarrow Ce^{k\cdot 0} = 100$$

$$C = 100$$

$\frac{1}{2}$-life means the amount of time it takes $P(t)$ to be reduced by $\frac{1}{2}$.

ie $\quad$ $\frac{1}{2}$-life is 1590 yrs $\Rightarrow$ $e^{k\cdot(1590)} = \frac{1}{2}$ $\qquad$ t in yrs

$$k \cdot 1590 = \ln\left(\tfrac{1}{2}\right)$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{1590} = -\frac{\ln 2}{1590}$$

Now need to find t such that

$$P(t) = 30$$

$$Ce^{kt} = 30$$

$$100\, e^{-\frac{\ln 2}{1590}\cdot t} = 30$$

$$e^{-\frac{\ln 2}{1590}\cdot t} = \frac{3}{10}$$

$$-\frac{\ln 2}{1590}\cdot t = \ln\left(\frac{3}{10}\right)$$

$$t = \frac{-\ln\left(\frac{3}{10}\right)}{\ln(2)} \cdot 1590 \approx 2672 \text{ yrs}$$

Related Rates

Ex Air being pumped into a (spherical) balloon such that the volume is increasing by $50\ \mathrm{cm}^3/\mathrm{s}$.
How fast is the radius of the balloon increasing when $r = 10$ cm?

Know $\frac{dV}{dt}$ $\quad$ V = volume $\qquad$ Want $\frac{dr}{dt}$ $\quad$ r = radius

Use the relation: $\quad V = \frac{4}{3}\pi r^3$

Apply $\frac{d}{dt}$ to both sides: $\quad \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$

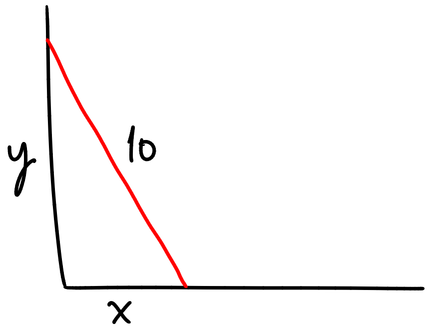
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad \left(\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} \right)$$

Now plug in what we know at particular moment:

$$50 = 4\pi (10)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{50}{4\pi \cdot 100} = \frac{1}{8\pi} \approx 0.040 \text{ cm/s}$$

Ex



10-ft ladder leaning against wall.
Bottom of ladder moves away from wall at 10 ft/s.
How fast does top of ladder move down the wall, when it is 2ft from the ground?

x in ft
t in s

Know $\frac{dx}{dt} = 10$ Want $\frac{dy}{dt}$.

Use $x^2 + y^2 = 10^2$.

Apply $\frac{d}{dt}$ to both sides: $\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(10^2)$

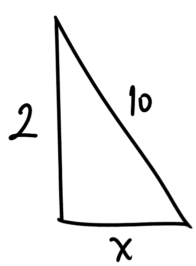
$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

Plug in: $\frac{dx}{dt} = 10$

$$20x + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{20x}{2y} = -\frac{10x}{y}$$

To find x and y:



$x^2 + 2^2 = 10^2$
$x = \sqrt{96} = \sqrt{16 \cdot 6} = 4\sqrt{6}$
$y = 2$

so $\frac{dy}{dt} = -\frac{10 \cdot 4\sqrt{6}}{2} = \underline{-20\sqrt{6} \text{ ft/s}}$

Ex $x = x(t)$ $\quad y = \sqrt{2x+1}$ $\quad$ If $\frac{dx}{dt} = 3$, what is $\frac{dy}{dt}$ when $x = 4$?
$y = y(t)$

① Directly: $\frac{dy}{dt} = \frac{1}{2} \frac{1}{\sqrt{2x+1}} \cdot 2 \cdot \frac{dx}{dt}$

$= \frac{1}{\sqrt{2x+1}} \frac{dx}{dt}$ $\quad$ plug in: $\frac{1}{\sqrt{2 \cdot 4 + 1}} \cdot 3 = \frac{1}{3} \cdot 3 = 1$

so $\frac{dy}{dt} = 1$

② Write $y^2 = 2x+1$, then $2y \cdot \frac{dy}{dt} = 2 \cdot \frac{dx}{dt}$

$\frac{dy}{dt} = \frac{1}{y} \cdot \frac{dx}{dt} = \frac{1}{\sqrt{2 \cdot 4 + 1}} \cdot 3 = 1$

Ex Gas in a box

Boyle's Law: $PV = C$ $\quad$ P = pressure of gas
V = volume of gas

Say $V = 400 \text{ cm}^3$

$P = 80$ kPa, decreasing at 10 kPa/min

At what rate is V increasing? $\quad$ Know $\frac{dP}{dt} = -10$ $\quad$ want $\frac{dV}{dt}$

$$PV = C$$

$$\frac{d}{dt}(PV) = \frac{d}{dt}(C)$$

$$\frac{dP}{dt} \cdot V + P \cdot \frac{dV}{dt} = 0$$

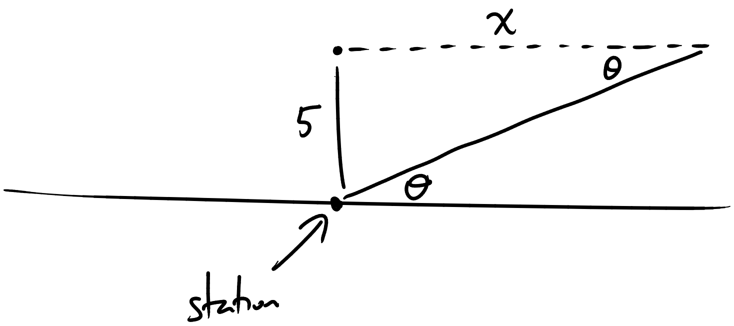
$$P \frac{dV}{dt} = -V \frac{dP}{dt}$$

$$\frac{dV}{dt} = -\frac{V}{P} \frac{dP}{dt}$$

$\frac{dP}{dt} = -10$ $\quad$ $P = 80$
$V = 400$

$$\frac{dV}{dt} = -\frac{400}{80} \cdot (-10) = \underline{50 \ cm^3/min}$$

Ex A plane flies at altitude 5km directly over a tracking station.



When angle of elev $\theta = \pi/3$ rad

and $\frac{d\theta}{dt} = -\pi/6$ rad/min

how fast is the plane moving?

Speed of plane $= \frac{dx}{dt}$ We know: $\theta, \frac{d\theta}{dt}$

Relation: $\tan\theta = \frac{5}{x}$ $\left(\longrightarrow \sqrt{3} = \frac{5}{x} \quad x = \frac{5}{\sqrt{3}}\right)$

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{5}{x^2}\frac{dx}{dt}$$

plug in: $\sec\theta = 2$, $x = 5/\sqrt{3}$, $\frac{d\theta}{dt} = -\pi/6$

so $$4 \cdot \left(-\frac{\pi}{6}\right) = -\frac{5}{\left(\frac{25}{3}\right)}\frac{dx}{dt}$$

$$-\frac{2\pi}{3} = -\frac{15}{25}\frac{dx}{dt}$$

$$\frac{2\pi}{3} = \frac{3}{5}\frac{dx}{dt}$$

$$\frac{10\pi}{9} = \frac{dx}{dt}$$

$\approx$

$\underline{3.49 \ km/min}$