Lecture 11

6 Oct 2015

Midterm 1 average ~ 87%

next HW due Fri night (Sat morning) 3am

office hr today 4–5:30

please let me borrow your notes from Lecture 10

Last time: implicit diff.

derivatives of logs and inverse trig functions

Exponential growth and decay

Suppose we have a function $P(t)$ such that $\frac{dP}{dt}$ is proportional to $P(t)$

ie $\frac{dP}{dt} = k \cdot P(t)$ for some constant k.

Then, what can $P(t)$ be?

$$P(t) = e^{kt} \text{ has } \frac{dP}{dt} = k \cdot e^{kt} = k \cdot P(t) \quad \checkmark$$

$$P(t) = Ce^{kt} \text{ works too: } \frac{dP}{dt} = C \cdot k \cdot e^{kt} = k \cdot (Ce^{kt}) = k \cdot P(t) \quad \checkmark$$

And this is the only function obeying that equation:

If $\frac{dP}{dt} = k \cdot P(t)$, **then** $P(t) = C \cdot e^{kt}$ for some C.

Ex
1. population growth under constant conditions $(k>0)$
2. radioactive decay $(k<0)$
3. compound interest $(k>0)$

Ex A population of bacteria grows from 1 g at 2pm to 15 g at 5pm.

What will be the mass of bacteria at 10pm?

$P(t) = Ce^{kt}$ $\qquad$ t = time past 2pm, in hours

Need to find C, k. $\qquad$ P in grams

Know: $P(0) = 1$

i.e. $C \cdot e^{k \cdot 0} = 1$, i.e. $C = 1$

$P(3) = 15$

i.e. $C \cdot e^{k \cdot 3} = 15$

$1 \cdot e^{3k} = 15$

$3k = \ln(15)$

$k = \ln(15)/3$

Thus $P(8) = C \cdot e^{k \cdot 8} = 1 \cdot e^{\frac{\ln(15)}{3} \cdot 8} = e^{\frac{8}{3}\ln(15)} = 15^{8/3}$

≈ 1368 g

Ex The $\frac{1}{2}$-life of radium-226 is 1590 yrs.

Suppose we have 100 mg of radium-226.

When will it be reduced to 30 mg by radioactive decay?

$P(t) = Ce^{kt}$. $\quad P(0) = 100$ $\qquad$ (P in mg)

ie $C \cdot e^{k \cdot 0} = 100$

ie $C = 100$

(t in yrs)

What is k? $\qquad e^{k \cdot (1590)} = \frac{1}{2}$ $\qquad$ (half life is 1590 yrs)

$k \cdot 1590 = \ln\left(\frac{1}{2}\right)$

$$k = \frac{\ln(\frac{1}{2})}{1590} = \frac{-\ln 2}{1590}$$

For which t will we have $P(t) = 30$?

$$Ce^{kt} = 30$$

$$100\, e^{-\frac{\ln 2}{1590} \cdot t} = 30$$

$$e^{-\frac{\ln 2}{1590} t} = \frac{3}{10}$$

$$-\frac{\ln 2}{1590} t = \ln \frac{3}{10}$$

$$t = \frac{-\ln \frac{3}{10}}{\ln 2} \cdot 1590 = \frac{\ln \frac{10}{3}}{\ln 2} \cdot 1590$$

$$\approx 2672 \text{ yrs}$$

Related Rates

Ex Air being pumped into a spherical balloon at $50\ \text{cm}^3/\text{s}$.
How fast is the radius of the balloon increasing when the radius is 10 cm?

Volume $V(t)$ in cm^3
time t in s

Know: $\frac{dV}{dt} = 50$

Radius $r(t)$ in cm

Want: $\frac{dr}{dt}$

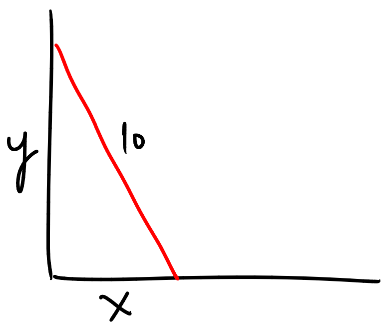
$V = \frac{4}{3}\pi r^3$ and do $\frac{d}{dt}$ of both sides:

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

so $\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{1}{4\pi r^2}$ plug in: $\frac{dV}{dt} = 50$ $r = 10$

$$\frac{dr}{dt} = 50 \cdot \frac{1}{4\pi \cdot 10^2} = \frac{1}{8\pi} \approx 0.040 \text{ cm/s}$$

Ex 10-ft ladder leaning against wall. Bottom of ladder moves away from wall at 1 ft/s. How fast does top of ladder move down the wall, when top is 2 ft from the ground?



Want $\frac{dy}{dt}$. Know $\frac{dx}{dt} = 1$ $\left(\begin{matrix} x \text{ in ft} \\ t \text{ in s} \end{matrix}\right)$

$$x^2 + y^2 = 10^2$$

Take $\frac{d}{dt}$ of both sides: $\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(10^2)$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2y\frac{dy}{dt} = -2x\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

Plug in: $\frac{dx}{dt} = 1$, $y = 2$, $x = \sqrt{10^2 - 2^2} = \sqrt{96} = 4\sqrt{6}$

so $\frac{dy}{dt} = -\frac{4\sqrt{6}}{2} \cdot 1 = \underline{\underline{-2\sqrt{6}}}$ ft/s

Ex Box of gas at constant temperature

Boyle's Law: $PV = C$ P = pressure V = volume

Say $V = 400 \text{ cm}^3$, $P = 80$ kPa, P <u>decreasing</u> at 10 kPa/min.

At what rate is V increasing?

$PV = C$ apply $\frac{d}{dt}$ to both sides

$$\frac{d}{dt}(PV) = \frac{d}{dt}(C)$$

$$\frac{dP}{dt} \cdot V + P \cdot \frac{dV}{dt} = 0$$

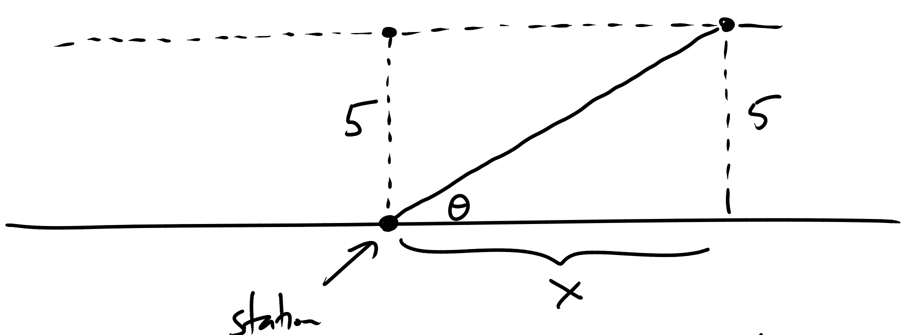
$$\frac{dV}{dt} = -\frac{V}{P}\frac{dP}{dt}$$

$$= -\frac{400}{80} \cdot (-10)$$

V in cm^3 t in min
P in kPa

$$= 50 \ \frac{cm^3}{min}$$

Ex Plane flying at altitude 5km goes directly over radar station.



When angle of elevation $\theta = \pi/3$ radians and $\frac{d\theta}{dt} = -\frac{\pi}{6}$ rad/min how fast is the plane moving?

This means: what is $\frac{dx}{dt}$?

Relation between x and θ:

$$\tan\theta = \frac{5}{x}$$

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{5}{x}\right)$$

$$\sec^2\theta \ \frac{d\theta}{dt} = -\frac{5}{x^2}\frac{dx}{dt}$$

Now plug in: $\frac{d\theta}{dt} = -\pi/6$, $\sec\theta = 2$, $x = \frac{5}{\tan\theta} = \frac{5}{\sqrt{3}}$

$(\theta = \pi/3)$

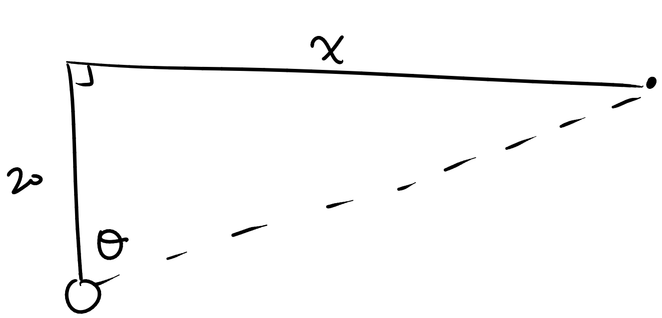
so $$2^2 \cdot \left(-\frac{\pi}{6}\right) = -\frac{5}{\left(\frac{5}{\sqrt{3}}\right)^2} \frac{dx}{dt}$$

$$-\frac{2\pi}{3} = -\frac{5}{25/3} \frac{dx}{dt}$$

$$\frac{2\pi}{3} = \frac{15}{25}\frac{dx}{dt} = \frac{3}{5}\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{10\pi}{9}\ \text{km/min} \approx 3.49\ \text{km/min}$$

<u>Ex</u>



Man walks along a wall at 4 ft/s.
Searchlight 20 ft from wall.

When the man is 15 ft from the point on the wall <u>closest</u> to the light, at what rate is the light rotating?

$\frac{dx}{dt} = 4$ (x in ft) (t in s)

want to find $\frac{d\theta}{dt}$.

<u>use</u> $x = 20\tan\theta$

and plug in $x = 15$ at the end...

should get $\frac{d\theta}{dt} \approx 0.128$ rad/s.