Lecture 12

8 Oct 2015

HW06 due Sat 3am
HW07 due Wed 3am

Regarding HW06:

(A) $\frac{d}{dx} \ln\left(\frac{(x+4)^3}{(x-2)^2}\right)$

Hard way: chain rule — $\frac{(x-2)^2}{(x+4)^3} \frac{d}{dx}\left(\frac{(x+4)^3}{(x-2)^2}\right) = \cdots$ (complicated)

$= \cdots$ (lots of simplifying)

$= \cdots$

$= \frac{3}{x+4} - \frac{2}{x-2}$

Easy way: $\ln\left(\frac{(x+4)^3}{(x-2)^2}\right) = \ln(x+4)^3 - \ln(x-2)^2$

$= 3\ln(x+4) - 2\ln(x-2)$

So $\frac{d}{dx}\ln\left(\frac{(x+4)^3}{(x-2)^2}\right) = 3\frac{d}{dx}\ln(x+4) - 2\frac{d}{dx}\ln(x-2)$

$= 3\cdot\frac{1}{x+4} - 2\cdot\frac{1}{x-2}$

$= \frac{3}{x+4} - \frac{2}{x-2}$

(B) $\frac{d}{dx}\left(\frac{x}{4}\right) = \frac{d}{dx}\left(\frac{1}{4}\cdot x\right) = \frac{1}{4}$

— could do this by Quotient Rule but that's a waste of time.

(C) How to simplify $\sin(\tan^{-1}x)$? (assuming $x>0$)

Let $\theta = \tan^{-1}x$. Then we want $\sin\theta$.

$\tan\theta = x$ (and $0<\theta<\frac{\pi}{2}$)

Algebraic way: find trig identities that relate $\sin\theta$ to $\tan\theta$

$$\tan\theta = x$$
$$\tan^2\theta = x^2$$
$$\sec^2\theta - 1 = x^2$$
$$\sec^2\theta = x^2+1$$
$$\cos^2\theta = \frac{1}{x^2+1}$$

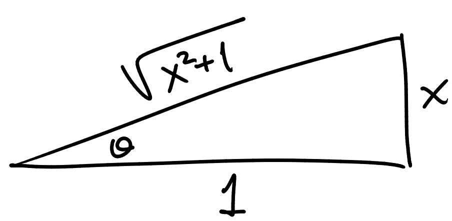
$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$$1 - \sin^2\theta = \frac{1}{x^2+1}$$
$$\sin^2\theta = 1 - \frac{1}{x^2+1}$$
$$\sin^2\theta = \frac{(x^2+1)-1}{x^2+1} = \frac{x^2}{x^2+1}$$
$$\sin\theta = \frac{x}{\sqrt{x^2+1}}$$

So, $\sin(\tan^{-1}x) = \dfrac{x}{\sqrt{x^2+1}}$

Geometric way: $\tan\theta = x$ want $\sin\theta$



$\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{x}{1} = x$

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{x}{\sqrt{x^2+1}}$

So, $\sin(\tan^{-1}x) = \dfrac{x}{\sqrt{x^2+1}}$

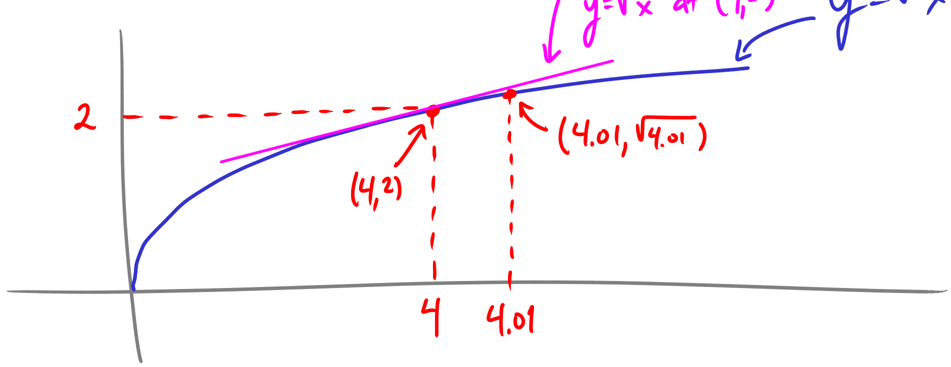
Today: another use of derivatives

Linear Approximation

What is $\sqrt{4}$? $\sqrt{4} = 2$

What is $\sqrt{4.01}$? "a little bigger than 2"

Let's try to estimate:



Tangent line:

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$. At $x=4$: $\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$ ← slope of tangent line

Tangent line is line thru (4,2) with slope $\frac{1}{4}$:

$$y-2 = \frac{1}{4}(x-4)$$
$$y = 2+\frac{1}{4}(x-4)$$

Plug in $x=4.01$: $y = 2+\frac{1}{4}(4.01-4)$
$= 2+\frac{1}{4}(0.01)$
$= 2.0025$

So, 2.0025 is our estimate of $\sqrt{4.01}$

and actually $\sqrt{4.01} = 2.0024984\ldots$ — pretty good accuracy!

Similarly: estimate $\sqrt{4.03}$. Already know the tangent line at (4,2):

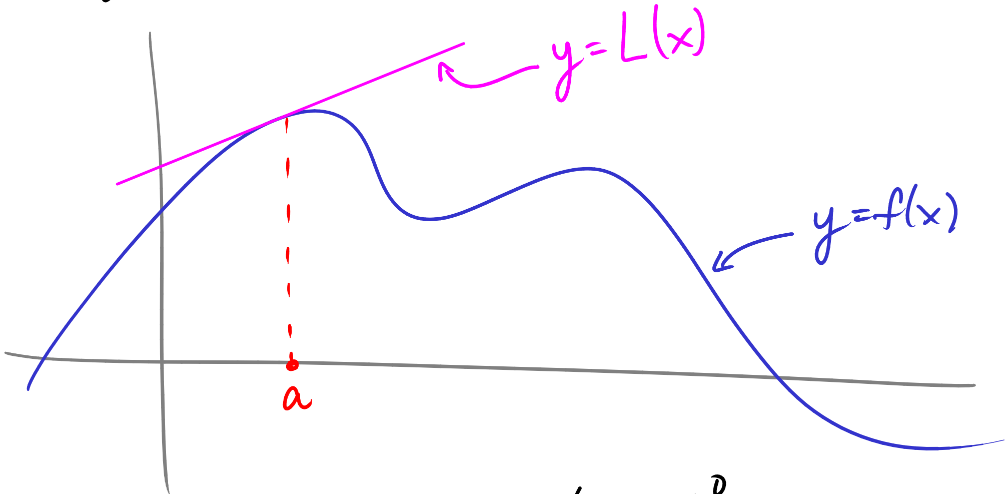
$$y = 2+\frac{1}{4}(x-4)$$

Plug in $x=4.03$: $y = 2+\frac{1}{4}(4.03-4) = 2+\frac{1}{4}(0.03)$
$= 2.0075$

and actually $\sqrt{4.03} = 2.0074859\ldots$

x	$2+\frac{1}{4}(x-4)$	$\sqrt{x}$
4	2	2
4.01	2.0025	2.002498...
4.03	2.0075	2.007486...
6	2.5	2.4495...
8	3	2.8284...
3.99	1.9975	1.997...

Generally:



$y=L(x)$ is the tangent line to $y=f(x)$ at $(a,f(a))$.

ie $y=L(x)$ is the line thru $(a,f(a))$ with slope $f'(a)=m$.

ie $y-f(a)=m(x-a)$

$L(x)-f(a)=f'(a)\cdot(x-a)$

$L(x)=f(a)+f'(a)\cdot(x-a)$

Note: $L(a)=f(a)+\cancel{f'(a)(a-a)}^{\,0}=f(a)$

Call $L(x)$ the linear approximation to $f(x)$ at a.

Ex Estimate $\cos\left(\frac{\pi}{2}+0.02\right)$.

$f(x)=\cos(x)$, $a=\pi/2$.

$L(x)=f(a)+f'(a)(x-a)$

$f(a)=\cos(\pi/2)=0$

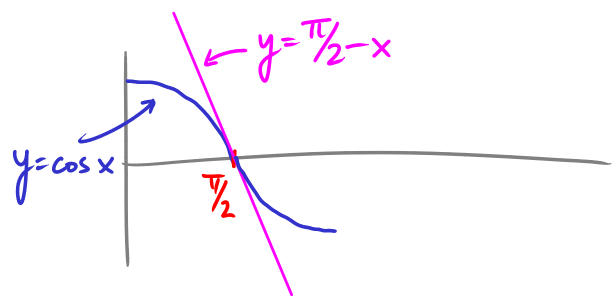
$f'(x)=-\sin(x)$

$f'(a)=-\sin(\pi/2)=-1$.

plug in $f(a)$ and $f'(a)$:

$L(x)=0+(-1)(x-\pi/2)$

$=\pi/2-x$



plug in $x=\pi/2+0.02$:

$L(\pi/2+0.02)=\pi/2-(\pi/2+0.02)$

$=-0.02$

So, $\cos\left(\frac{\pi}{2}+0.02\right)\approx \underline{\underline{-0.02}}$

Ex Estimate $\sqrt[3]{25}$.

$f(x)=\sqrt[3]{x}$. Take $a=27$.

$L(x)=f(a)+f'(a)(x-a)$

$f(a)=\sqrt[3]{27}=3$

$f'(x)=\frac{d}{dx}x^{1/3}=\frac{1}{3}x^{-2/3}$

$f'(a)=\frac{1}{3}(27^{-2/3})$

$f(a) = 3$

$f'(a) = \frac{1}{27}$

so $L(x) = 3 + \frac{1}{27}(x-27)$

$= \frac{1}{3}\left(27^{1/3}\right)^{-2}$

$= \frac{1}{3}(3)^{-2}$

$= \frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27}$

so $\sqrt[3]{25} = f(25) \approx L(25) = 3 + \frac{1}{27}(25-27)$

$= 3 - \frac{2}{27} = \frac{79}{27}$

Ex $\sqrt[3]{26} \approx 3 + \frac{1}{27}(26-27)$

$= 3 - \frac{1}{27} = \frac{80}{27}$

Ex $\sqrt[3]{28} \approx 3 + \frac{1}{27} = \frac{82}{27}$

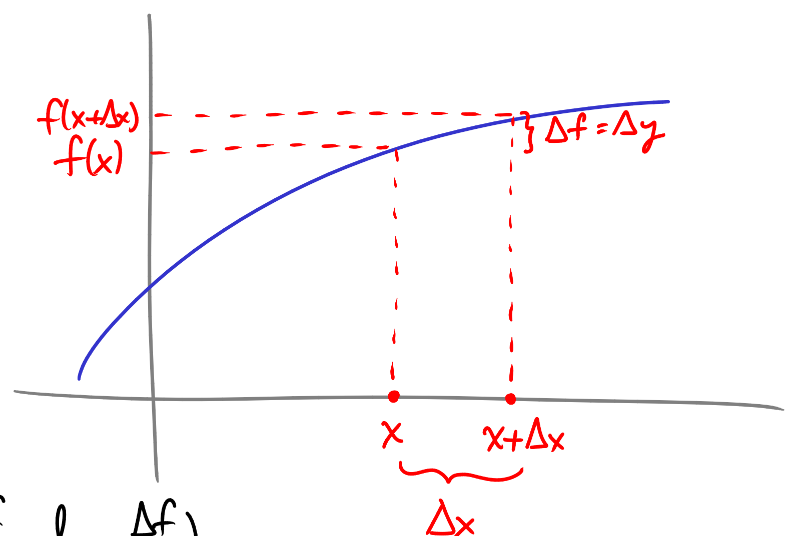
$\sqrt[3]{29} \approx \frac{83}{27}$

$\vdots$

Another way to think about this:

$f(x+\Delta x) = f(x) + \Delta f$

$f(x+\Delta x) = f(x) + \frac{\Delta f}{\Delta x} \cdot \Delta x$



If Δx is very small, then

$\frac{\Delta f}{\Delta x}$ is about $\frac{df}{dx}$

$\left(\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}\right)$

So $f(x+\Delta x) \approx f(x) + \frac{df}{dx} \cdot \Delta x$

Differential of f: write "$df = \frac{df}{dx} \cdot dx$" as an analog of $\Delta f \approx \frac{df}{dx} \Delta x$

(think of dx as infinitesimal variation of x,
df as infinitesimal variation of f)

e.g. $d(\tan x) = \frac{d(\tan x)}{dx} \cdot dx = \sec^2 x \, dx$

$$d(\tan x) = \sec^2 x \, dx$$

(this <u>means</u> $\Delta(\tan x) \approx \sec^2 x \, \Delta x$ for small Δx)

<u>Ex</u> Estimate $\ln(1.2)$.

$x = 1$ $\quad$ $\ln(x) = 0$
$\Delta x = 0.2$

$$\ln(x + \Delta x) = \ln(x) + \Delta(\ln x)$$

$d(\ln x) = \frac{1}{x} dx$, so $\Delta(\ln x) \approx \frac{1}{x} \Delta x$

so, $\ln(x + \Delta x) \approx \ln(x) + \frac{1}{x} \Delta x$

$$\ln(1 + 0.2) \approx \ln(1) + \frac{1}{1}(0.2)$$
$$= 0 + 0.2$$
$$= 0.2$$