Lecture 13

13 Oct 2015

My office hours today: 2pm–3pm

Last time: related rates

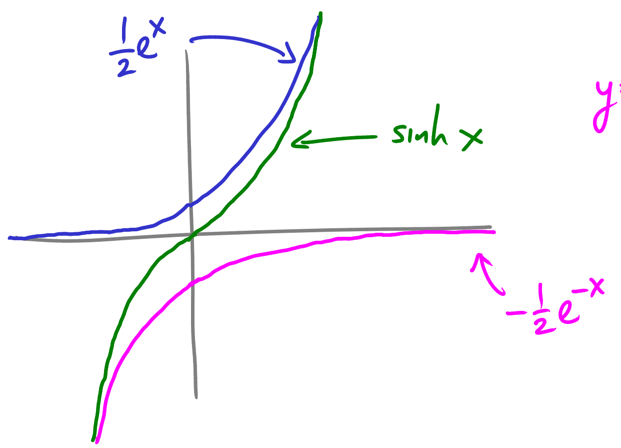
Hyperbolic functions

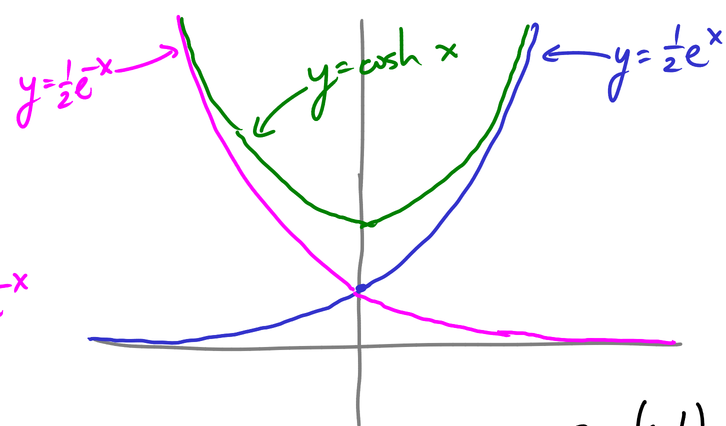
Hyperbolic functions are "cousins" of the usual trig functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$





range(sinh) is $(-\infty, \infty)$.

range(cosh) is $[1, \infty)$.

Remark: $y = \cosh x$ is the shape of a freely-hanging heavy cord (like a telephone wire). ("catenary")

Hyperbolic identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Why? E.g. $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ $\qquad$ $\sinh(-x) = \dfrac{e^{-x} - e^{-(-x)}}{2}$

$$= \frac{-e^x + e^{-x}}{2}$$

$$= -\left(\frac{e^x - e^{-x}}{2}\right) = -\sinh(x) \checkmark$$

E.g. $\cosh^2 x - \sinh^2 x$

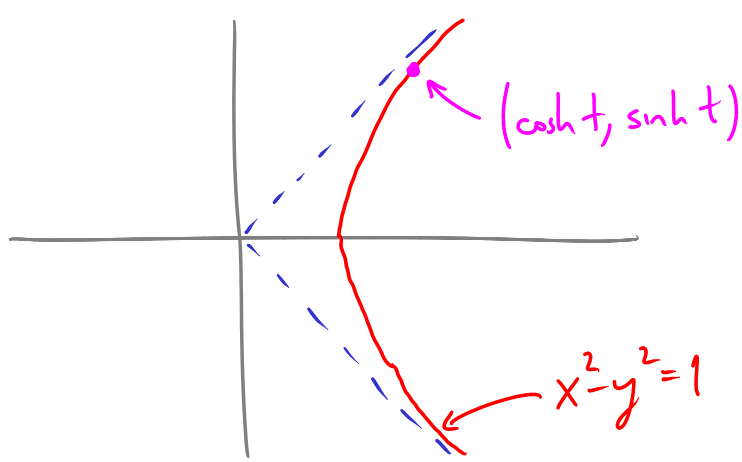
$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4}$$

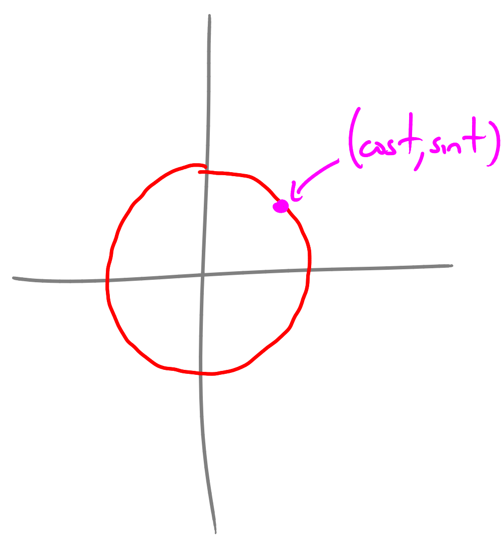
$$= \frac{e^{2x} + 2 \cdot e^x \cdot e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2 \cdot e^x \cdot e^{-x} + e^{-2x}}{4}$$

$$= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4} = \frac{4}{4} = 1.$$

So, $\cosh^2 t - \sinh^2 t = 1$: if we let $\begin{array}{l} x = \cosh t \\ y = \sinh t \end{array}$ then $x^2 - y^2 = 1$



(cf.



Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

why? $\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$

$$= \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2}$$

$$= \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

<u>Ex</u> $\frac{d}{dx}(8\cosh 2x) = 16 \sinh 2x$

<u>Remark</u> Similarity to usual trig functions can be explained by:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

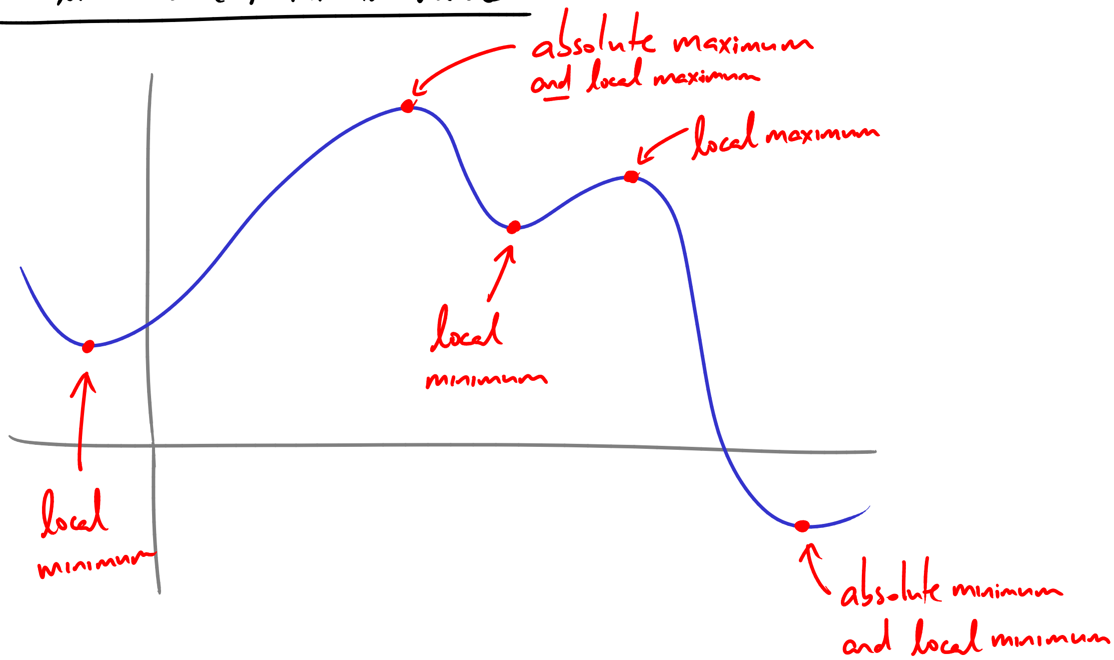
$$e^{ix} = \cos x + i \sin x$$

$$e^{ix} + e^{-ix} = (\cos x + i\sin x) + (\cos(-x) + i \sin(-x))$$

$$= (\cos x + i \sin x) + (\cos x - i \sin x)$$

$$= 2\cos x$$

Maximum and Minimum Values

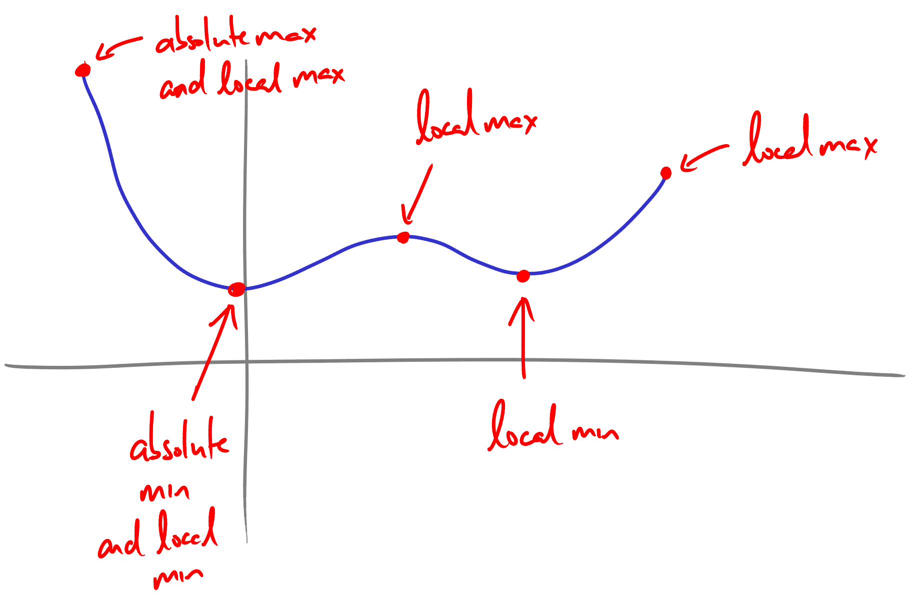


Say $f(c)$ is <u>absolute max</u> value of f if $f(c) \geq f(x)$ for every x in domain of f.

$f(c)$ is <u>absolute min</u> value of f if $f(c) \leq f(x)$ for every x in domain of f.

$f(c)$ is <u>local max</u> value of f if $f(c) \geq f(x)$ for every x near c.

$f(c)$ is <u>local min</u> value of f if $f(c) \leq f(x)$ for every x near c.

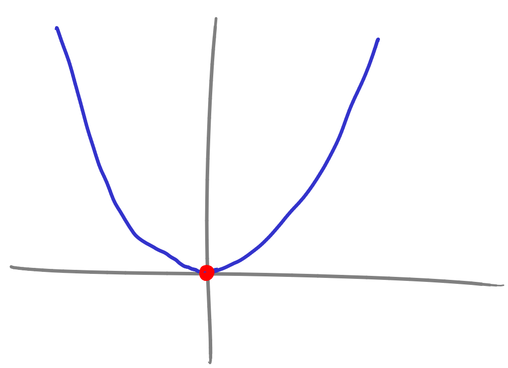


<u>Ex</u> $f(x) = x^2$ domain $(-\infty, \infty)$

absolute minimum: $x=0$ $f(x)=0$

local minimum: $x=0$ $f(x)=0$

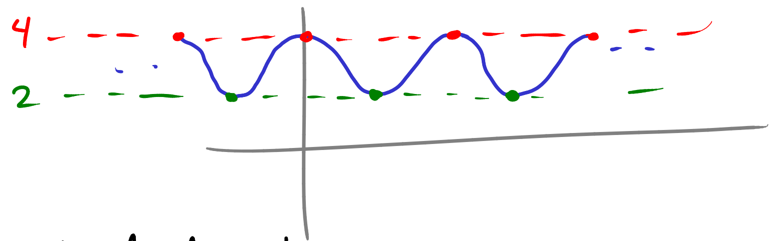
<u>no</u> local or absolute max



Ex $f(x) = 3 + \cos x$

(local and) absolute max: $f(x) = 4$ at $x = 0, 2\pi, -2\pi, 4\pi, -4\pi, \ldots$

(local and) absolute min: $f(x) = 2$ at $x = \pi, -\pi, 3\pi, -3\pi, \ldots$



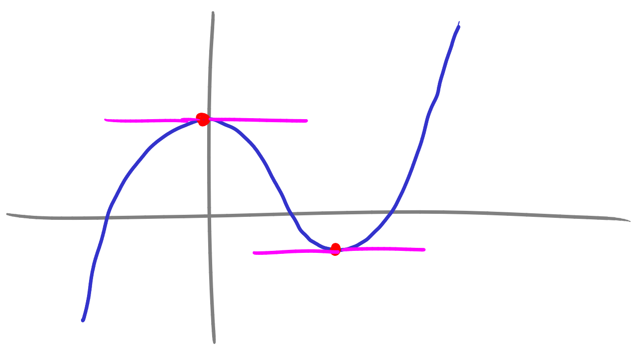
no other local max/min

Fact: If $f(x)$ has a local max or min at $x = c$,

and $f'(c)$ exists,

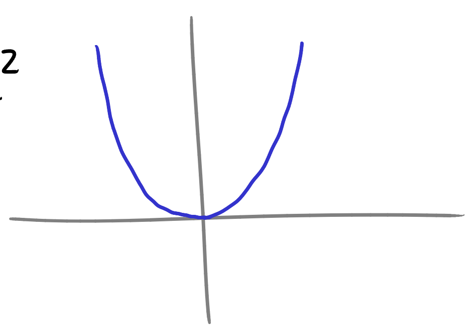
and the domain of f contains an interval around c (c isn't on the end of the domain)

Then, $f'(c) = 0$.



(ie: the places where the graph of $y = f(x)$ turns around and is differentiable are places where it has a horizontal tangent)

Ex $f(x) = x^2$



$f'(x) = 2x$

so $f'(x) = 0$ only at $x = 0$

and $f'(x)$ exists for all x;

so the only possible place for a local min/max is $x = 0$.

Indeed there is a local min at $x = 0$.

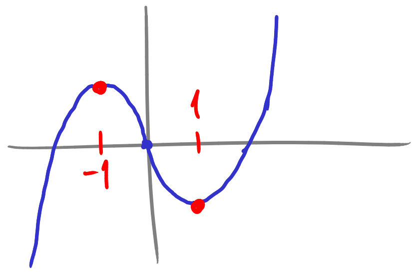
Ex $f(x) = x^3 - 3x$ $\quad f'(x) = 3x^2 - 3$ $\quad$ exists for all x

$f'(x) = 0$ only at $\quad 0 = 3x^2 - 3$

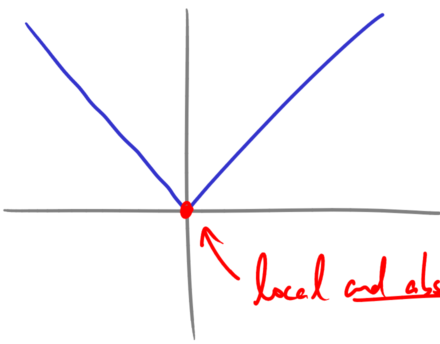
$0 = 3(x+1)(x-1)$

ie. $\quad x = 1$ or $x = -1$.

So the only possible local max/min for $f(x)$ are at $x = 1$, $x = -1$.



Ex $f(x) = |x|$



local and absolute min.

$f'(0)$ does not exist:

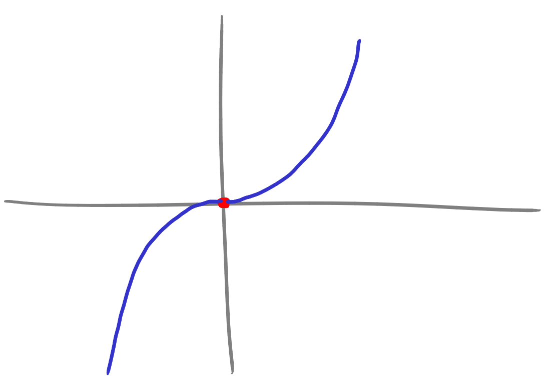
so this point could be a min or max.

(In this case it is.)

Ex $f(x) = x^3$

$f'(x) = 3x^2$ so $f'(x) = 0$ only at $x = 0$. And $f'(x)$ exists for all x.

So the only possible local min or max is $x = 0$.



But, in fact, here $x = 0$ is not a local max or min.

This $f(x)$ has no local max or mins

Strategy for finding absolute max/min for a function f with domain $[a,b]$:

1. find values of f at all "critical numbers":
   x where $f'(x)=0$ or $f'(x)$ DNE.

2. find values of $f(a)$, $f(b)$

3. take max, min values of f from this list.

Ex Find absolute max, min of

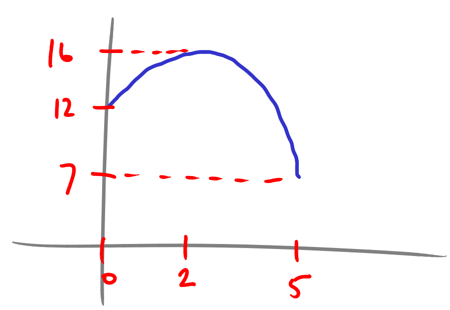
$$f(x) = 12+4x-x^2 \quad \text{on} \quad [0,5]. \qquad (0 \le x \le 5)$$

1. $f'(x)$ exists everywhere — no points where $f'(x)$ DNE.
   $f'(x) = 4-2x$
   so $f'(x)=0$ only at $x=2$. So only critical # is $x=2$.

   $f(2) = 12+8-4 = 16.$

2. $f(0) = 12+0-0 = 12.$
   $f(5) = 12+20-25 = 7.$

3. max is $f(2)=16.$
   min is $f(5)=7.$



Ex Find critical #'s of

$$f(x) = x^{-2}\ln x \quad \text{for} \quad 1 \le x \le 100.$$

$$f'(x) = -2x^{-3}\ln x + x^{-2}\cdot x^{-1}$$
$$= x^{-3}(-2\ln x + 1)$$

So $f'(x)=0$ only if $-2\ln x+1=0$

$$\ln x=\frac{1}{2}$$

$$x=e^{\frac{1}{2}}=\sqrt{e}.$$

So $x=\sqrt{e}$ is the only critical #.

$$f(1)=0$$

$$f(\sqrt{e})=\frac{1}{e}\ln(\sqrt{e})=\frac{1}{2e}$$

$$f(100)=\ln(100)/10000$$

So max value of f on $[1,100]$ is $\frac{1}{2e}$.