Lecture 15

20 Oct 2015

Midterm 2 next Thursday Oct 29

Format like the last one but probably a bit harder

Covers material from after last midterm, through Thursday's lecture (Lecture 16)

including the HW due next Wed Oct 28 (shorter)

Next lecture given by Jamie Pool

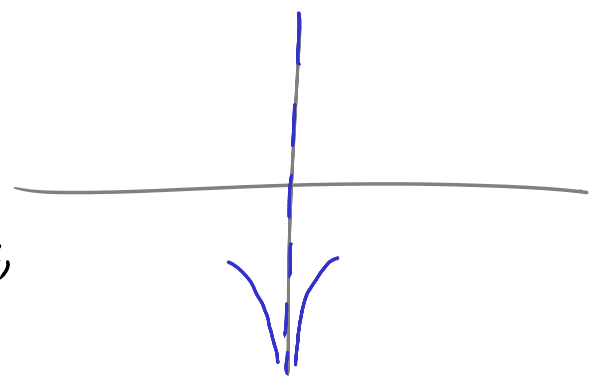
Last time: graphing using derivatives.

Ex Sketch the graph of $f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$.

Domain: all real #'s except $x=0$

$= (-\infty, 0) \cup (0, \infty)$

As $x \to 0$, $\lim_{x\to 0} f(x) = -\infty$



(why? one way: if x is very small, +ve or -ve,

$-\frac{1}{x^2}$ is huge and negative,

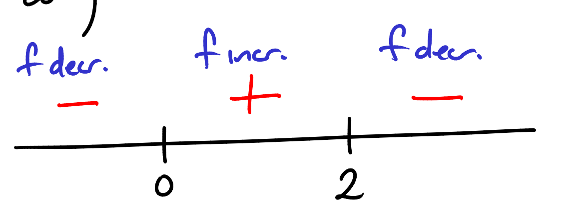
and much bigger than $\frac{1}{x}$

more formal way: $\lim_{x\to 0} 1 + \frac{1}{x} - \frac{1}{x^2} = \lim_{x\to 0} -\frac{1}{x^2}(1 - x - x^2)$

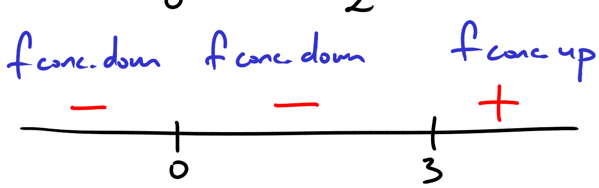
$= \lim_{x\to 0} -\frac{1}{x^2}$

$= -\infty$)

$$f'(x) = -\frac{1}{x^2} + \frac{2}{x^3} = \frac{1}{x^3}(2-x)$$



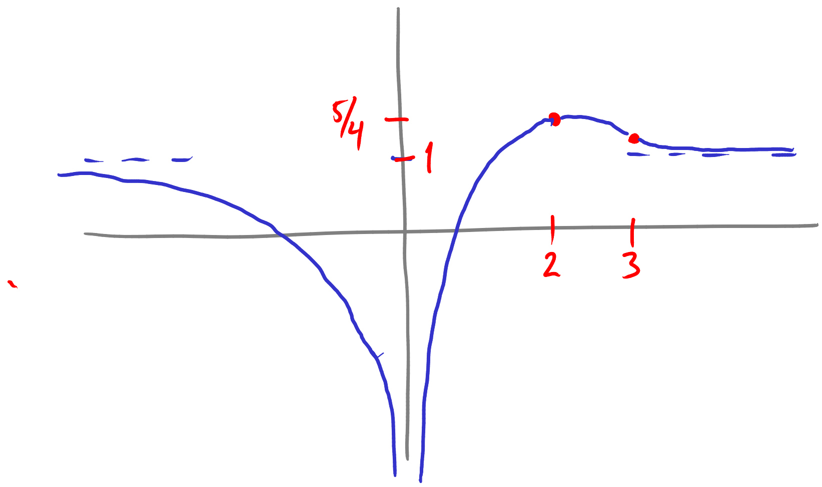
$$f''(x) = \frac{2}{x^3} - \frac{6}{x^4} = \frac{1}{x^4}(2x-6)$$



$f(2) = 1 + \frac{1}{2} - \frac{1}{4} = \frac{5}{4}$

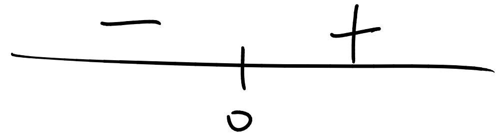
$f(3) = 1 + \frac{1}{3} - \frac{1}{9} = \frac{11}{9}$

$f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$ $\lim_{x \to \infty} f(x) = 1$ $\lim_{x \to -\infty} f(x) = 1$

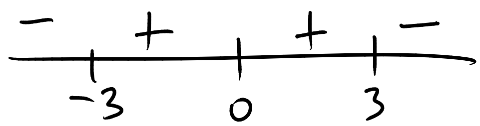


<u>Ex</u> Sketch graph of $y = \ln(x^4 + 27)$.

$f' = \frac{4x^3}{x^4+27}$



$f'' = \ldots$

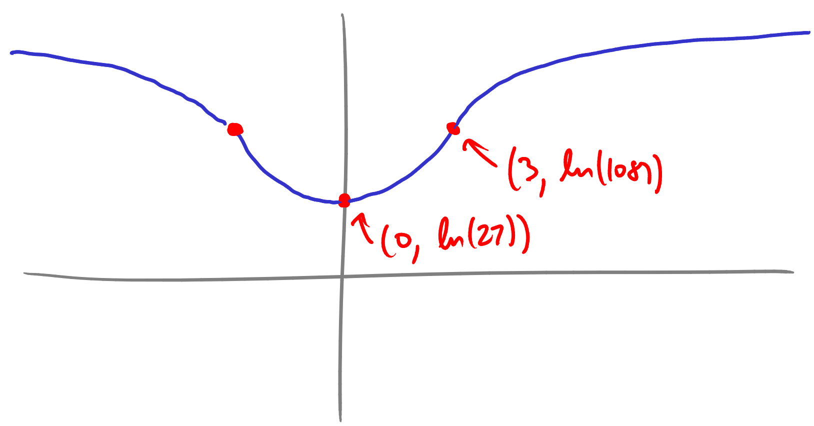


Draw it yourself!

$f(0) = \ln(27)$

$f(3) = \ln(108)$

$f(3) > f(0) > 0$



L'Hospital's Rule

We can use derivatives as a trick for calculating limits.

How to calculate $\lim_{x \to 1} \frac{\ln x}{x-1}$? Can't just plug in $x=1$: that would give $\frac{0}{0}$. And can't simplify to cancel anything.

Call this $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$ an "indeterminate form".

Fact Say $f(x)$ and $g(x)$ are differentiable near $x=a$ and $g'(x)\neq 0$ near $x=a$ (except at $x=a$).

Say $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$,

or $\lim_{x\to a} f(x)=\pm\infty$ and $\lim_{x\to a} g(x)=\pm\infty$.

Then, $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ if that limit exists, or is $\pm\infty$.

Ex $\lim_{x\to 1}\frac{\ln x}{x-1}$. Plug in $x=1$, get $\frac{0}{0}$. So, L'Hospital's rule applies.
Differentiate top and bottom. get $\lim_{x\to 1}\frac{1/x}{1}=\lim_{x\to 1}\frac{1}{x}=\underline{1}$.

Can also use this for limits as $x\to\pm\infty$.

Ex $\lim_{x\to\infty}\frac{e^x}{x^2}=?$ Taking limits on top, bottom separately gives $\frac{\infty}{\infty}$.
So, L'Hospital's rule applies:

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{e^x}{2x}.$$ Again $\frac{\infty}{\infty}$. Use L'H rule again:

$$=\lim_{x\to\infty}\frac{e^x}{2}=+\infty.$$

Ex $\lim_{x\to\infty}\frac{e^x}{3^x}=?$ L'H: $\lim_{x\to\infty}\frac{e^x}{(\ln 3)\cdot 3^x}$ — doesn't help.

but $\lim_{x\to\infty}\frac{e^x}{3^x}=\lim_{x\to\infty}\left(\frac{e}{3}\right)^x=0$ since $\frac{e}{3}<1$.

Ex $\lim_{x\to 0}\frac{\tan x-x}{x^3}=?$ Plug in $x=0$: $\frac{0-0}{0}=\frac{0}{0}$ so L'H applies.

$\to=\lim_{x\to 0}\frac{\sec^2 x-1}{3x^2}$ Plug in $x=0$: $\frac{1-1}{3\cdot 0}=\frac{0}{0}$.

Could do L'H again. Or, use trig identity:

$$= \lim_{x\to 0} \frac{\tan^2 x}{3x^2} = \frac{1}{3}\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^2 = \frac{1}{3}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\left(\frac{1}{\cos x}\right)^2$$

$$= \frac{1}{3}\,(1)^2(1^2) = \underline{\underline{\frac{1}{3}}}$$

Ex $\lim_{x\to 0^-} \frac{\sin x}{1+\cos x}$ Plug in $x=0$: $\frac{0}{2} = \underline{\underline{0}}.$

Can't use L'Hospital here: not $\frac{0}{0}$ or $\frac{\infty}{\infty}$!

(would get $\frac{\cos x}{-\sin x} = -\infty$, wrong!)

Ex $\lim_{x\to 0^+} x\ln x$ plug in $x=0$: $0\cdot(-\infty)$, another "indeterminate form"

Trick: $x\ln x = \frac{\ln x}{\left(\frac{1}{x}\right)}$ plug in $x=0$: $\frac{(-\infty)}{\infty}$ so can use L'Hospital

$$\lim_{x\to 0}\frac{\ln x}{\left(\frac{1}{x}\right)} = \lim_{x\to 0}\frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0}\left(\frac{-x^2}{x}\right) = \lim_{x\to 0}(-x) = \underline{\underline{0}}.$$

Ex $\lim_{x\to 0}\frac{x+\sin x}{x+\cos x} = \frac{0+0}{0+1} = 0$ (by plugging in!)

<u>Indeterminate Powers</u>

$\lim_{x\to a} f(x)^{g(x)} = ?$ (e.g. $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x} = ?$)

Sometimes can do this by plugging in, sometimes not.

We may get something like "∞^0" "0^0" "1^∞" — all indeterminate forms.

A trick: try taking log.

i.e. if $L = \lim_{x\to a} f(x)^{g(x)}$ then $\ln L = \ln\left(\lim_{x\to a} f(x)^{g(x)}\right)$

$$= \lim_{x\to a}\left(\ln f(x)^{g(x)}\right)$$

$$= \lim_{x\to a} \left(g(x)\cdot \ln f(x)\right)$$

<u>Ex</u> $\lim_{x\to 0^+} x^x = ?$

Let $L = \lim_{x\to 0^+} x^x$

Then $\ln L = \lim_{x\to 0^+} \ln(x^x)$

$$= \lim_{x\to 0^+} x \ln x$$

$$= 0 \quad \text{(by previous example)}$$

So $\ln L = 0$

so $L = e^0 = 1$, i.e. $\underline{\underline{\lim_{x\to 0^+} x^x = 1}}$

<u>Ex</u> $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = ?$

$L = \lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$

$\ln L = \lim_{x\to 0^+} \cot x \cdot \ln(1+\sin 4x)$ plug in $\to \infty \cdot 0$

$$= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x}$$ plug in $\to \frac{0}{0}$

use L'Hi $= \lim_{x\to 0^+} \dfrac{\frac{1}{1+\sin 4x}\cdot 4\cos 4x}{\sec^2 x}$

$$= \frac{\frac{1}{1+0}\cdot 4 \cdot 1}{1} = 4$$

$\ln L = 4$

$\underline{\underline{L = e^4}}$