Lecture 15

20 Oct 2015

Midterm 2 next Thursday Oct 29

Format just like last one, but probably a bit harder

Covers material from after the last midterm, through Thursday's lecture (Lecture 16)

including HW due Wed Oct 28 (shorter)

Next lecture given by Jamie Pool

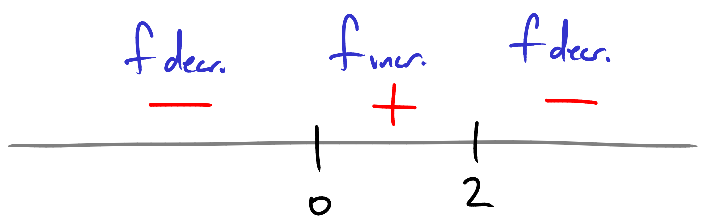
Last time: graphing using derivatives

Ex Sketch the graph of $f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$.

Domain: all x except $x=0$,

i.e. $(-\infty, 0) \cup (0, \infty)$

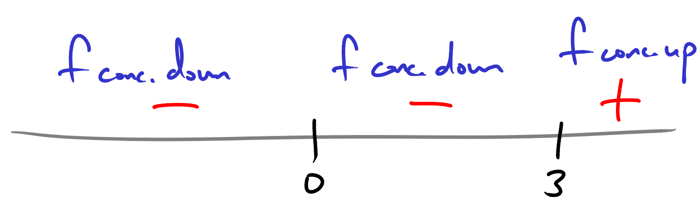
$$f'(x) = -\frac{1}{x^2} + \frac{2}{x^3} = \frac{1}{x^3}(-x+2)$$



$f'(2) = 0$ ie $x=2$ is critical pt

$x=0$ isn't in the domain

$$f''(x) = \frac{2}{x^3} - \frac{6}{x^4} = \frac{1}{x^4}(2x-6)$$



$$f(2) = 1 + \frac{1}{2} - \frac{1}{4} = \frac{5}{4}$$

$$f(3) = 1 + \frac{1}{3} - \frac{1}{9} = \frac{11}{9}$$

Asymptotes:

(1) what happens as $x \to 0$?

$$\lim_{x\to 0} 1 + \underbrace{\frac{1}{x}}_{+\infty} \underbrace{- \frac{1}{x^2}}_{-\infty} = -\infty, \text{ why?}$$

informally: $-\frac{1}{x^2}$ goes to $-\infty$ much faster than $+\frac{1}{x}$ goes to $+\infty$

formally: $1+\frac{1}{x}-\frac{1}{x^2} = -\frac{1}{x^2}\left(-x^2-x+1\right)$

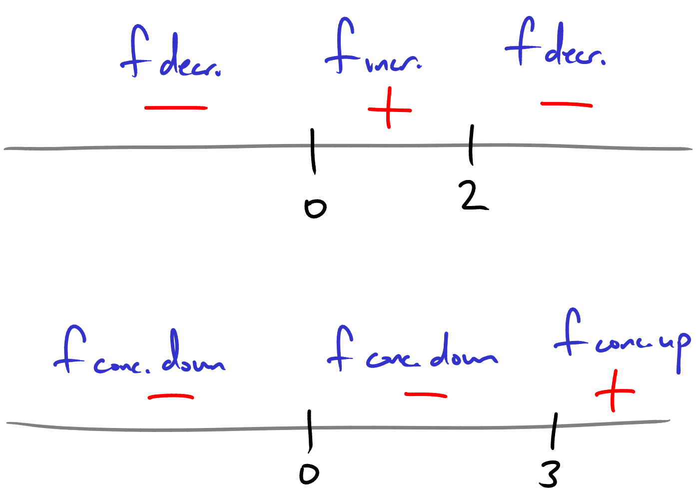
so as $x \to 0$, $\lim\limits_{x\to 0} -\frac{1}{x^2}(-x^2-x+1) = \lim\limits_{x\to 0}\left(-\frac{1}{x^2}\cdot 1\right) = -\infty$

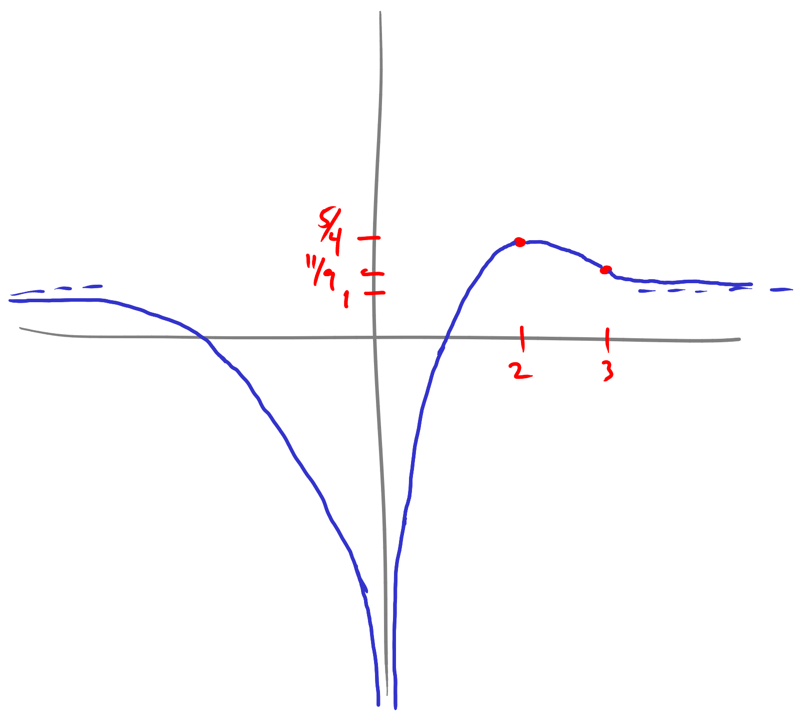
so, $\lim\limits_{x \to 0} f(x) = -\infty$. vertical asymptote at $x=0$.

(2) what happens as $x \to \pm\infty$?

$\lim\limits_{x\to\infty} 1+\frac{1}{x}-\frac{1}{x^2} = 1$, $\lim\limits_{x\to-\infty} 1+\frac{1}{x}-\frac{1}{x^2} = 1$.

horizontal asymp at $x=1$ in both directions.



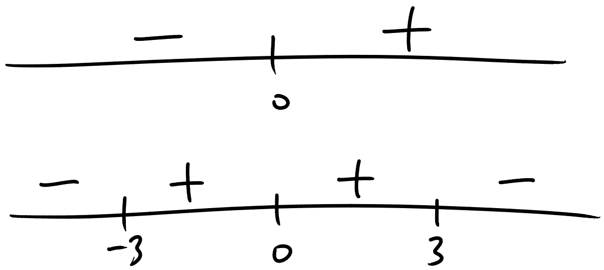


Ex Sketch graph of $f(x) = \ln(x^4+27)$.

$f' = \frac{4x^3}{x^4+27}$

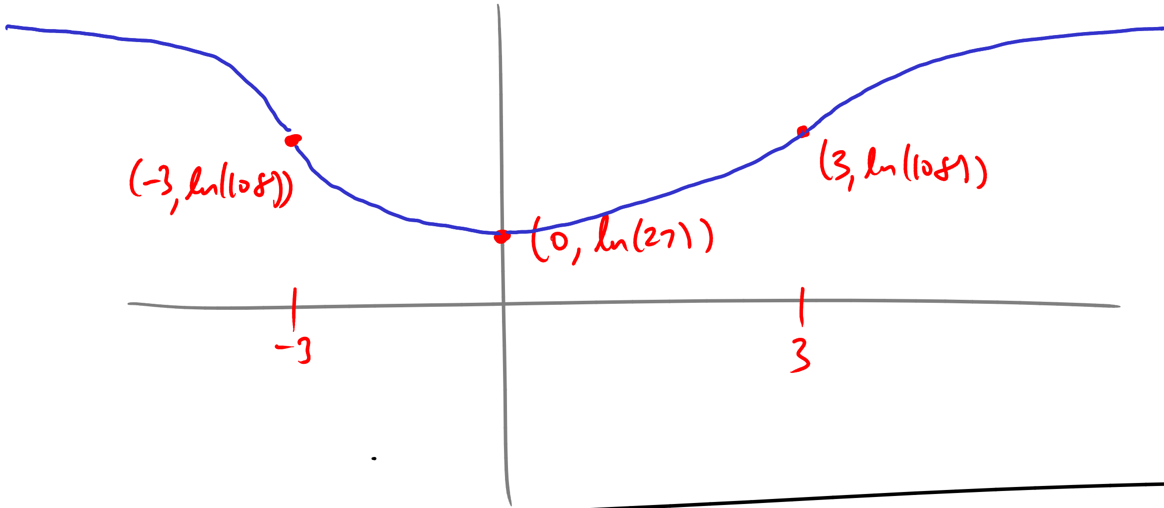
$f'' = (\cdots)$

All x in domain.



$\ln(27) \approx 3$

$\ln(108) \approx 4.5$



L'Hospital's Rule

We can use derivatives to calculate limits.

How to calculate $\lim_{x\to 1} \frac{\ln x}{x-1}$? Plugging in gives $\frac{0}{0}$ — no help.

No way to simplify to "cancel the 0's." ("indeterminate form")

Fact Say $f(x)$ and $g(x)$ are differentiable near $x=a$ and $g'(x) \neq 0$ near $x=a$ (except maybe at $x=a$)

$\frac{f}{g}$

Say $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$,

or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$.

Then, $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, if that limit exists or is $\pm\infty$.

("L'Hospital's rule")

Ex $\lim_{x\to 1} \frac{\ln x}{x-1}$. Plug in $x=1$: get $\frac{0}{0}$. $\longrightarrow$ L'Hospital's rule applies.

So, $\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = \lim_{x\to 1} \frac{1}{x} = \frac{1}{1} = 1.$

Can also use this for limits as $x \to \pm\infty$, one-sided limits...

Ex $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2} = ?$ Taking limits on top, bottom separately gives $\frac{\infty}{\infty}$.

$\to$ L'Hospital's Rule applies.

So, $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2} = \lim\limits_{x\to\infty} \dfrac{e^x}{2x}$. Try again: still $\frac{\infty}{\infty}$.

$\to$ L'Hospital's Rule still applies.

So, $\ldots = \lim\limits_{x\to\infty} \dfrac{e^x}{2} = \underline{\underline{\infty}}$.

Ex $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^3} = ?$ Plug in $x=0$: $\dfrac{0-0}{0} = \dfrac{0}{0} \to$ L'H Rule applies.

So, $= \dfrac{\sec^2 x - 1}{3x^2}$. Plug in $x=0$: $\dfrac{1-1}{0} = \dfrac{0}{0} \to$ Could use L'H Rule again.

Or, use trig identity: $= \dfrac{\tan^2 x}{3x^2}$.

$$\lim_{x\to 0} \frac{\tan^2 x}{3x^2} = \frac{1}{3}\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^2$$

$$= \frac{1}{3}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\left(\frac{1}{\cos x}\right)^2$$

$$= \frac{1}{3}(1)^2\left(\frac{1}{1}\right)^2 = \underline{\underline{\frac{1}{3}}}$$

Ex $\lim\limits_{x\to 0} \dfrac{\sin x}{1+\cos x} = \dfrac{0}{2} = 0$.

(don't and $\underline{\underline{\text{can't}}}$, use L'H Rule for this, because it's not $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ after plugging in)

Ex $\lim\limits_{x\to 0^+} x \ln x = ?$ Taking limits separately gives "$0 \cdot (-\infty)$" — no help

Trick: $\underline{\text{rewrite}}$ $x \ln x = \dfrac{\ln x}{\left(\frac{1}{x}\right)}$. Taking lim as $x \to 0$ now: $\dfrac{-\infty}{\infty}$

can use L'H Rule.

$$\lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} = \lim_{x\to 0^+} \frac{1/x}{\left(-1/x^2\right)} = \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+}(-x) = 0.$$

Indeterminate Powers

$\lim_{x\to a} f(x)^{g(x)} = ?$ e.g. $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = ?$

Sometimes can do this by "plugging in", sometimes not.

We may get indeterminate forms like ∞^0, 0^0, 1^∞

A trick: try taking log.

i.e. if $L = \lim_{x\to a} f(x)^{g(x)}$ then

$$\begin{aligned} \ln L &= \ln\left(\lim_{x\to a} f(x)^{g(x)}\right) \\ &= \lim_{x\to a} \ln\left(f(x)^{g(x)}\right) \\ &= \lim_{x\to a} \left(g(x)\ln f(x)\right). \end{aligned}$$

Ex $\lim_{x\to 0^+} x^x = ?$

Let $L = \lim_{x\to 0^+} x^x$

Then
$$\begin{aligned} \ln L &= \lim_{x\to 0^+} \ln(x^x) \\ &= \lim_{x\to 0^+} x\ln x \\ &= 0 \quad \text{(last example)} \end{aligned}$$

$\ln L = 0$

i.e. $L = e^0 = 1$ So $\lim_{x\to 0^+} x^x = 1.$

Ex $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = ?$ 1^∞ — indeterminate

$L = \lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$

$$\ln L = \lim_{x\to 0^+} (\cot x)\cdot \ln(1+\sin 4x). \qquad \infty\cdot 0$$

$$= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\frac{1}{\cot x}}$$

$$= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} \qquad \frac{0}{0}$$

use L'H: $$= \lim_{x\to 0^+} \frac{\frac{1}{1+\sin 4x}\cdot 4\cos 4x}{\sec^2 x} = \frac{\frac{1}{1+0}\cdot 4\cdot 1}{1} = 4$$

$$\ln L = 4$$

$$L = e^4 \quad \text{ie} \quad \lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = \underline{\underline{e^4}}.$$