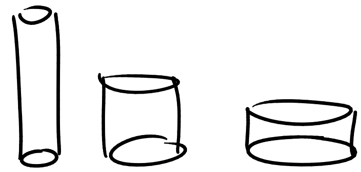
Lecture 17

26 Oct 2015

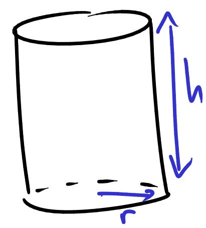
Midterm 2 Thursday.

My office hr today 4:00–5:30 as usual.

Last time: <u>optimization</u>



Ex A cylindrical can without a top is to hold V cm^3 of liquid. What are the dimensions for the can which <u>minimize</u> the cost of metal for making the can?



$$V = \pi r^2 h$$

Want to minimize $A = \underset{\text{bottom}}{\pi r^2} + \underset{\text{sides}}{2\pi r h}$

Eliminate the variable h: $V = \pi r^2 h$

$$\rightarrow h = \frac{V}{\pi r^2}$$

so $A = \pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right) = \pi r^2 + \frac{2V}{r}$

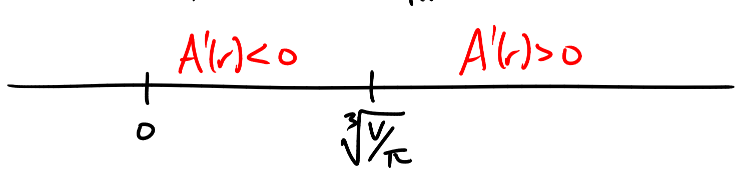
Want to find <u>absolute minimum</u> allowed value for A:

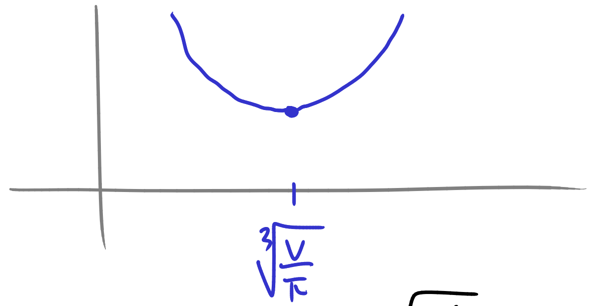
A is a function of one variable r domain $(0,\infty)$

$$A'(r) = 2\pi r - \frac{2V}{r^2}$$

$$= 2\pi r\left(1 - \frac{V}{\pi r^3}\right)$$

$A'(r) = 0$ just at $1 - \frac{V}{\pi r^3} = 0$ $\quad \frac{V}{\pi r^3} = 1$ $\quad \frac{V}{\pi} = r^3$ $\quad r = \sqrt[3]{\frac{V}{\pi}}$

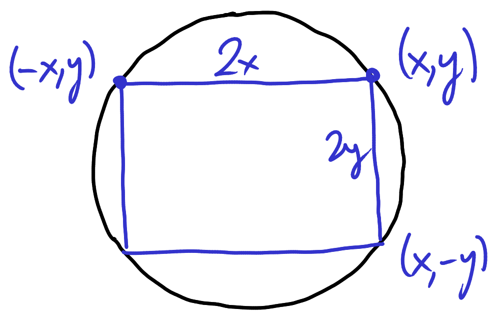




So, the absolute minimum occurs at $r = \sqrt[3]{\frac{V}{\pi}}$

Then $h = \frac{V}{\pi r^2} = \frac{V}{\pi \cdot \left(\frac{V}{\pi}\right)^{2/3}} = \frac{V/\pi}{(V/\pi)^{2/3}} = \sqrt[3]{\frac{V}{\pi}}$ also.

<u>Ex</u> Find the largest possible area for a rectangle inscribed in a circle of radius 1.



$$x^2+y^2=1$$
$$A=(2x)(2y)=4xy$$

One approach: <u>eliminate</u> a variable, say y, by $y=\sqrt{1-x^2}$

then write $A = 4xy = 4x\sqrt{1-x^2}$

now A is a function of one variable x. Domain: $x \in [0,1]$.

To find max for A: ① find critical pts

$$A'(x) = 4\sqrt{1-x^2} + 4x\left(\frac{-2x}{2\sqrt{1-x^2}}\right)$$
$$= 4\left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right)$$
$$= 4\left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}}\right)$$
$$= \frac{4}{\sqrt{1-x^2}}(1-2x^2)$$

so $A'(x)=0$ just if $2x^2=1$, i.e. $x^2=\frac{1}{2}$, i.e. $x=\frac{1}{\sqrt{2}}$.

Value at critical pt: $A\left(\frac{1}{\sqrt{2}}\right) = 4\left(\frac{1}{\sqrt{2}}\right)\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2} = 4\left(\frac{1}{\sqrt{2}}\right)\sqrt{\frac{1}{2}} = 2.$

(2) values at endpoints. $A(0) = 0\sqrt{1} = 0$

$A(1) = 1 \cdot \sqrt{0} = 0$

(3) max occurs at $A\left(\frac{1}{\sqrt{2}}\right) = 2$. ie at $x = \frac{1}{\sqrt{2}}$. Then $y = \sqrt{1-x^2}$

$$= \sqrt{1-\tfrac{1}{2}}$$

$$= \frac{1}{\sqrt{2}}.$$

$\longrightarrow$ best possible area obtained by taking a <u>square</u> with side length $2x = \frac{2}{\sqrt{2}} = \sqrt{2}$.

Second approach: $A = 4xy$ $\qquad x^2+y^2=1$

Differentiate: $\dfrac{dA}{dx} = 4y + 4x\dfrac{dy}{dx}$ $\qquad 2x + 2y\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{x}{y}$$

So $\dfrac{dA}{dx} = 4y + 4x\left(-\dfrac{x}{y}\right) = 4y - 4\dfrac{x^2}{y}$

$\dfrac{dA}{dx} = 0$ if $4y = 4\dfrac{x^2}{y}$

$y^2 = x^2$

$\underline{\underline{y = x}}$ (both $x, y > 0$)

and $x^2+y^2=1$, so $2x^2=1$ i.e. $x = \frac{1}{\sqrt{2}}$ — (faster way of getting the critical point!)

<u>Antiderivatives</u>

- $f(x) = x^2 \leadsto f'(x) = 2x$
- $f(x) = \sin(x^3) \leadsto f'(x) = 3x^2\cos(x^3)$

Suppose we want to "go backwards":

$$f \underset{\text{antiderivative}}{\overset{\text{derivative}}{\rightleftarrows}} f'$$

We say $F(x)$ is an <u>antiderivative</u> for $f(x)$ if $F'(x) = f(x)$.

<u>Ex</u> $f(x) = x$ has antiderivative $\frac{1}{2}x^2$

<u>or</u> $\frac{1}{2}x^2 + 2$

<u>or</u> $\frac{1}{2}x^2 + 37\pi$

$\vdots$

To get all possible antiderivatives of $f(x)$, first find <u>one</u> antiderivative, and then add an arbitrary constant. (usually called C)

<u>Ex</u> $f(x) = \cos x$ has general antiderivative $F(x) = \sin x + C$

$f(x) = x^n$ has general antiderivative $F(x) = \dfrac{x^{n+1}}{n+1} + C$ ← <u>if</u> $n \neq -1$

$$\left(\frac{d}{dx} F(x) = \frac{1}{n+1} \cdot (n+1) x^n = x^n\right)$$

$f(x) = \dfrac{1}{1+x^2}$ has general antiderivative $F(x) = \tan^{-1} x + C$

$f(x) = \dfrac{1}{x}$ has general antiderivative $F(x) = \ln x + C$

Build more complicated examples from these:

$$f(x) = 9x^2 + 6x^{3/2} - \frac{2}{x^4} + \cos 2x$$

has general antideriv.

$$F(x) = 3x^3 + 6\left(\frac{1}{\left(\frac{5}{2}\right)} x^{5/2}\right) - 2\left(\frac{1}{(-3)} x^{-3}\right) + \frac{1}{2}\sin 2x + C$$

Sometimes we don't want the most general antiderivative, we want some specific one.

Ex What is the function $F(x)$ which has $F'(x) = 4x+7$

and $F(1) = 6$?

Since $F'(x) = 4x+7$

we have $F(x) = 2x^2+7x+C$;

and $F(1) = 6$, so

$$2(1^2)+7(1)+C = 6$$
$$9+C=6$$
$$C=-3$$

thus $\underline{F(x) = 2x^2+7x-3}$

Why care about antiderivatives?

Standard reason:

$$\begin{tikzcd}[column sep=huge]
\text{position } s(t) \arrow[r, bend left, "\text{derivative}"] & \text{velocity } v(t) \arrow[r, bend left, "\text{derivative}"] \arrow[l, bend left, "\text{antiderivative}"] & \text{acceleration } a(t) \arrow[l, bend left, "\text{antiderivative}"]
\end{tikzcd}$$

Ex A train accelerates with constant acceleration, $a(t) = 4 \text{ ft}/\text{s}^2$

At time $t=0$ it has velocity 100 ft/s.

How far does it go in 20 s?

$$a(t) = 4$$
$$v(0) = 100$$
$$s(0) = 0$$

t in s

Want $s(20)$.

s is antideriv. of v

v is antideriv. of a

$$v(t) = 4t + C$$

and $v(0) = 100$, so $4 \cdot (0) + C = 100$, i.e. $C = 100$

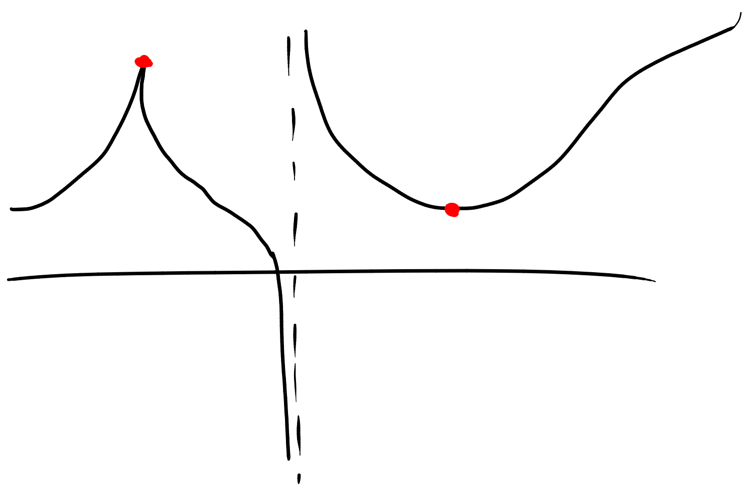
so $v(t) = 4t + 100$.

Then $s(t) = 2t^2 + 100t + D$

and $s(0) = 0$, so $0 + 0 + D = 0$, i.e. $D = 0$.

So $s(t) = 2t^2 + 100t$.

$s(20) = 2(20^2) + 100(20) = 800 + 2000 = \underline{\underline{2800 \text{ ft}}}$.



$$\lim_{x \to \infty} \frac{x}{3} \cdot \ln\left(\frac{x+6}{x}\right) \qquad \text{looks like} \quad \infty \cdot 0$$

$$= \lim_{x \to \infty} \frac{\ln\left(1 + \frac{6}{x}\right)}{\left(\frac{3}{x}\right)} \qquad \frac{0}{0}$$

$$= \lim_{x \to \infty} \frac{\left(\frac{1}{1+6/x}\right) \cdot \left(-\frac{6}{x^2}\right)}{\left(-3/x^2\right)} = 2 \lim_{x \to \infty} \frac{1}{1 + 6/x} = \frac{2}{}$$