Lecture 17

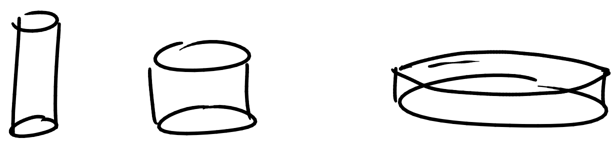
27 Oct 2015

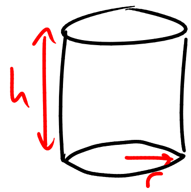
Midterm 2 Thu (next class)

My office hr today 4–5:30pm RLM 9.134

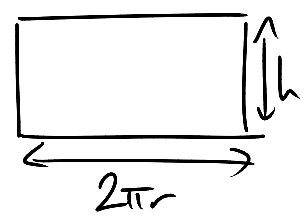
Last time: Optimization

Ex A cylindrical can without a top is to hold V cm^3 of liquid. What are the dimensions for the can which minimize the cost of metal?





$$V = \underbrace{\pi r^2}_{\text{area of base}} \underset{\substack{\uparrow \\ \text{height}}}{h}$$



Want to minimize surface area: $A = \underset{\substack{\uparrow \\ \text{base}}}{\pi r^2} + \underset{\substack{\uparrow \\ \text{sides}}}{2\pi r h}$

Eliminate h using our constraint:

$$V = \pi r^2 h$$

$$\frac{V}{\pi r^2} = h$$

Then $A = \pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right) = \pi r^2 + 2\frac{V}{r}$

function of one variable r, domain $(0,\infty)$.

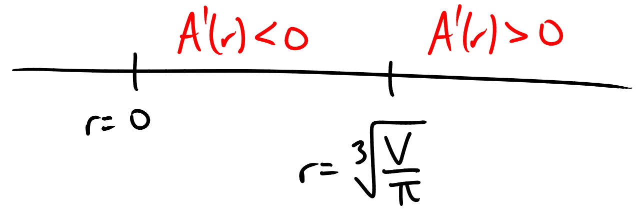
To find absolute minimum: look at $A'(r)$

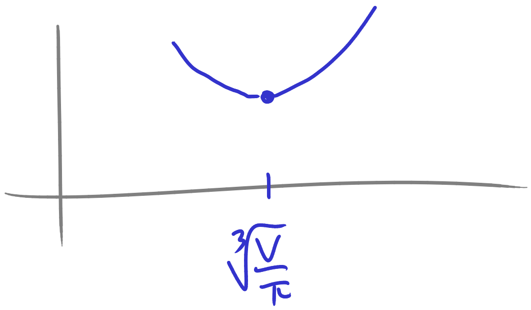
$$A'(r) = 2\pi r - 2\frac{V}{r^2}$$

$$A'(r) = 2\pi r\left(1 - \frac{V}{\pi r^3}\right)$$

$A'(r) = 0$ just if $1 - \frac{V}{\pi r^3} = 0$

$$\frac{V}{\pi r^3} = 1 \quad \frac{V}{\pi} = r^3 \quad \sqrt[3]{\frac{V}{\pi}} = r$$

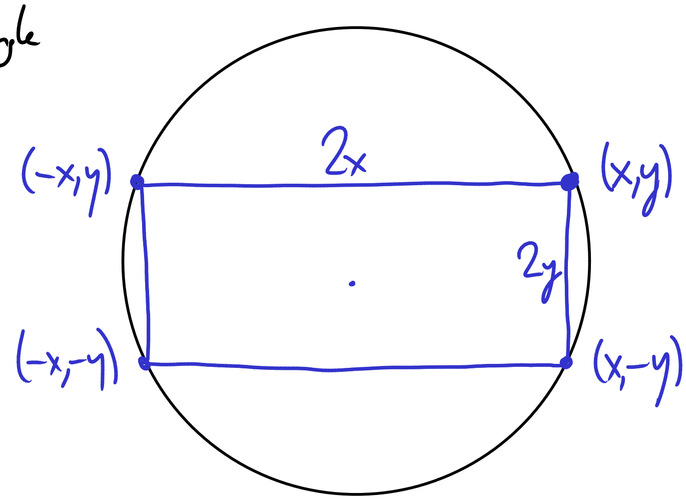




Thus the absolute minimum of $A(r)$ is attained at $r = \sqrt[3]{\frac{V}{\pi}}$.

$$h = \frac{V}{\pi r^2} = \frac{V}{\pi \cdot \left(\frac{V}{\pi}\right)^{2/3}} = \frac{V/\pi}{(V/\pi)^{2/3}} = \left(\frac{V}{\pi}\right)^{1/3} = \sqrt[3]{\frac{V}{\pi}}$$

Ex Find the largest area possible for a rectangle inscribed in a circle of radius 1.



$$x^2 + y^2 = 1$$

$$A = (2x)(2y) = 4xy$$

First approach: eliminate y

$$y = \sqrt{1-x^2}$$

then $A = 4x\sqrt{1-x^2}$

function of a single variable, $A(x)$, with domain $[0,1]$.

① find critical points: $A'(x) = 4\left(\sqrt{1-x^2} + x \frac{d}{dx}\sqrt{1-x^2}\right)$

$$= 4\left(\sqrt{1-x^2} + x\frac{(-2x)}{2\sqrt{1-x^2}}\right)$$

$$= 4\left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right)$$

$$= 4\left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}}\right)$$

$$= \frac{4}{\sqrt{1-x^2}}(1-2x^2)$$

so $A'(x)=0$ just if $1-2x^2=0$ ie $2x^2=1$

$$x^2 = 1/2$$

$$x = \frac{1}{\sqrt{2}}$$

(not $-\frac{1}{\sqrt{2}}$, this isn't in domain)

So $x=\frac{1}{\sqrt{2}}$ is the only critical pt.

$$A\left(\frac{1}{\sqrt{2}}\right) = 4\left(\frac{1}{\sqrt{2}}\right)\cdot\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= 4\left(\frac{1}{\sqrt{2}}\right)\cdot\sqrt{\frac{1}{2}} = 2.$$

Here $y=\sqrt{1-x^2}=\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}=x$

so this rectangle is a square, with side length $=\frac{1}{\sqrt{2}}$.

(2) check endpoints: $A(0)=4\cdot 0\cdot\sqrt{1}=0$

$A(1)=4\cdot 1\cdot\sqrt{0}=0.$

So maximum area is 2, obtained at $x=\frac{1}{\sqrt{2}}$, $y=\frac{1}{\sqrt{2}}$.

Alternate way: $A=4xy$ $\qquad x^2+y^2=1$

want to find places where $\frac{dA}{dx}=0$.

Differentiate both equations:

$$\frac{dA}{dx}=4y+4x\frac{dy}{dx} \qquad\qquad 2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

$$\rightarrow \quad \frac{dA}{dx} = 4y - 4\frac{x^2}{y}$$

So $\frac{dA}{dx} = 0$ means

$$0 = 4y - 4\frac{x^2}{y}$$

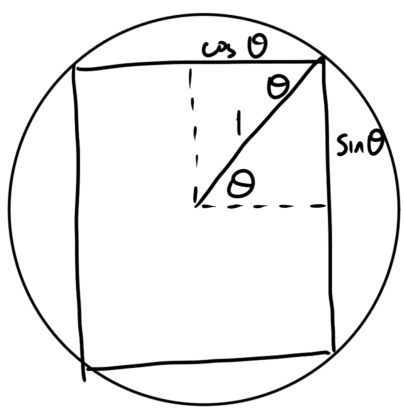
$$4y = 4\frac{x^2}{y}$$

$$y^2 = x^2$$

$$y = x \qquad (x, y > 0)$$

So the critical pt. is a rectangle which is a square.

$$x = y$$
$$x^2 + y^2 = 1$$
$$2x^2 = 1$$
$$x = \frac{1}{\sqrt{2}}.$$



$$A = 4\sin\theta\cos\theta$$

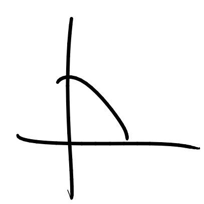
$$\frac{dA}{d\theta} = \cdots$$

or: $A = 2\sin 2\theta$

$$\frac{dA}{d\theta} = 4\cos 2\theta$$

so $\frac{dA}{d\theta} = 0$ at $\cos 2\theta = 0$

i.e. $\theta = \frac{\pi}{4}$



Antiderivatives

$$f(x) = x^2 \rightsquigarrow f'(x) = 2x$$

$$f(x) = \sin(x^3) \rightsquigarrow f'(x) = 3x^2\cos(x^3)$$

Suppose we want to "go backwards":

$$f \underset{\text{antiderivative}}{\overset{\text{derivative}}{\rightleftarrows}} f'$$

Any continuous function f has <u>many</u> antiderivatives!

<u>Ex</u>: $f(x) = x$ has antiderivatives

$$F(x) = \frac{1}{2}x^2$$
$$F(x) = \frac{1}{2}x^2 + 12$$
$$F(x) = \frac{1}{2}x^2 + 76\pi - 8$$
$$\vdots$$

To get <u>all possible</u> antideriv. for $f(x)$,
first find <u>one</u> antideriv, then add an arbitrary constant (usually called C)

<u>Ex</u>
- $f(x) = \cos x$ has general antiderivative $F(x) = \sin x + C$
- $f(x) = x^n$ has general " $F(x) = \dfrac{x^{n+1}}{n+1} + C$ <u>if $n \neq -1$!</u>

(e.g. $f(x) = x^4$ has antideriv. $F(x) = \frac{x^5}{5} + C$)

- $f(x) = 9x^2 + 6x^{3/2} - \dfrac{2}{x^4} + \cos 2x$ $\qquad \dfrac{1}{x^4} = x^{-4}$

has general antideriv.

$$F(x) = 9 \cdot \frac{x^3}{3} + 6 \cdot \frac{x^{5/2}}{(5/2)} - 2 \cdot \frac{x^{-3}}{-3} + \frac{1}{2}\sin 2x + C$$
$$= 3x^3 + \frac{12}{5}x^{5/2} + \frac{2}{3}x^{-3} + \frac{1}{2}\sin 2x + C$$

- $f(x) = x^{-1} = \dfrac{1}{x}$ has general antideriv. $\ln x + C$

Sometimes we want a <u>specific</u> antideriv:

<u>Ex</u> What is the function $F(x)$ such that $F'(x) = 4x + 7$ and $F(1) = 6$?

Since $F'(x) = 4x + 7$

have $F(x) = 4\left(\frac{x^2}{2}\right) + 7(x) + C$

$$= 2x^2 + 7x + C$$

and $F(1) = 6$, so $2(1^2) + 7(1) + C = 6$

$$9 + C = 6$$

$$C = -3$$

So $\underline{\underline{F(x) = 2x^2 + 7x - 3}}$

Why care about antideriv?

One reason:

$$\begin{tikzcd}[column sep=huge] \text{position } s(t) \arrow[r, bend left, "\text{derivative}"] & \text{velocity } v(t) \arrow[r, bend left, "\text{derivative}"] \arrow[l, bend left, "\text{antiderivative}"] & \text{acceleration } a(t) \arrow[l, bend left, "\text{antiderivative}"] \end{tikzcd}$$

Ex A train accelerates with constant accel. $a(t) = 4\ ft/s^2$.
At time $t=0$ it has velocity $v(t=0) = 100\ ft/s$
and position $s(t=0) = 0\ ft$.

How far does it go in $20s$? $s(t = 20s) = ?$

$a(t) = 4$

$$\left.\begin{array}{l} v(t) = 4t + C \\ v(t=0) = 100 \end{array}\right] \Rightarrow C = 100$$

So $v(t) = 4t + 100$

$$\left.\begin{array}{l} s(t) = 2t^2 + 100t + D \\ s(t=0) = 0 \end{array}\right] \Rightarrow D = 0$$

So $s(t) = 2t^2 + 100t$

$s(20) = 2(20^2) + 100 \cdot 20 = \underline{\underline{2800 \text{ ft}}}$

$f(x)$ A critical point of f is a point x s.t. x is in domain of f

and either $f'(x) = 0$

or $f'(x)$ DNE