Lecture 18

3 Nov 2015

Exam 2 average $\approx 81\%$

My office hr today 4–5:30 RLM 9.134

HW due Fri 3am

Last time: antiderivatives

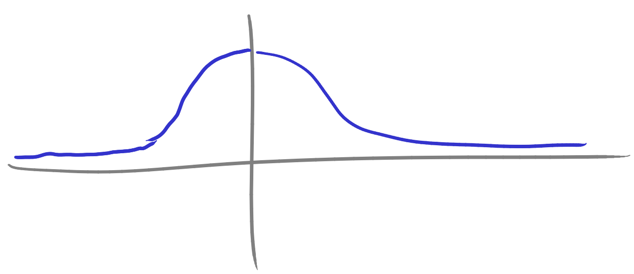
Ex $f(x) = x^4 + x$ has antideriv. $\frac{x^5}{5} + \frac{x^2}{2} + C$ for any constant C.

Because $\frac{d}{dx}\left(\frac{x^5}{5} + \frac{x^2}{2} + C\right) = x^4 + x$.

Remark Every continuous function has an antiderivative.

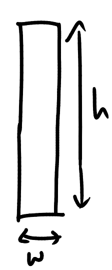
But, e.g. the antiderivative of $f(x) = e^{-x^2}$

cannot be written in terms of "elementary" functions $(+, \times, -, \div, \exp, \log, \sin, \cos, \sin^{-1}, \ldots)$



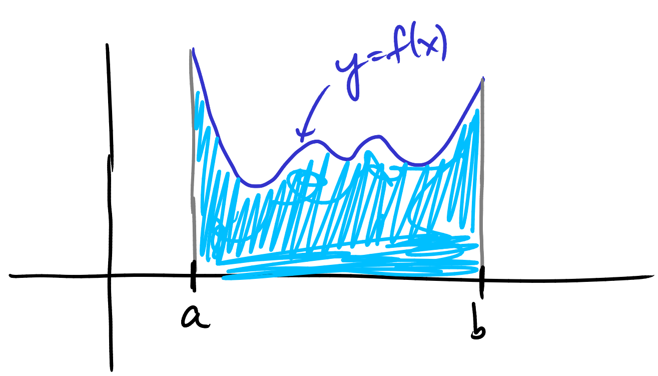
We give this antiderivative a new name: "error function"

Areas We all know areas of simple shapes



$A = w \cdot h$

How about more complicated shapes?

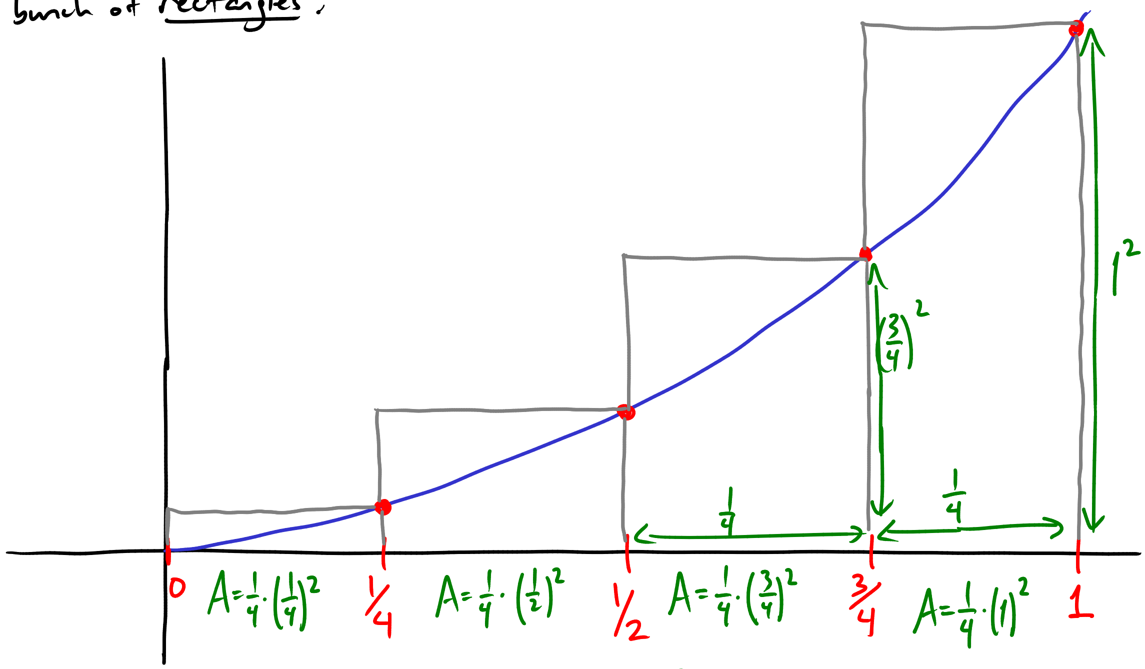


Let's try to estimate the area under graph of $y = f(x)$ over the x-axis between $x = a$ and $x = b$.

Ex Say $f(x) = x^2$.

Estimate the area of the region between $y = f(x)$ and the x-axis, and between $x = 0$ and $x = 1$.

Idea: approximate our region by a bunch of <u>rectangles</u>.



$$\text{total area of rectangles} = \frac{1}{4}\left[\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + (1)^2\right]$$

$$= \frac{15}{32}$$

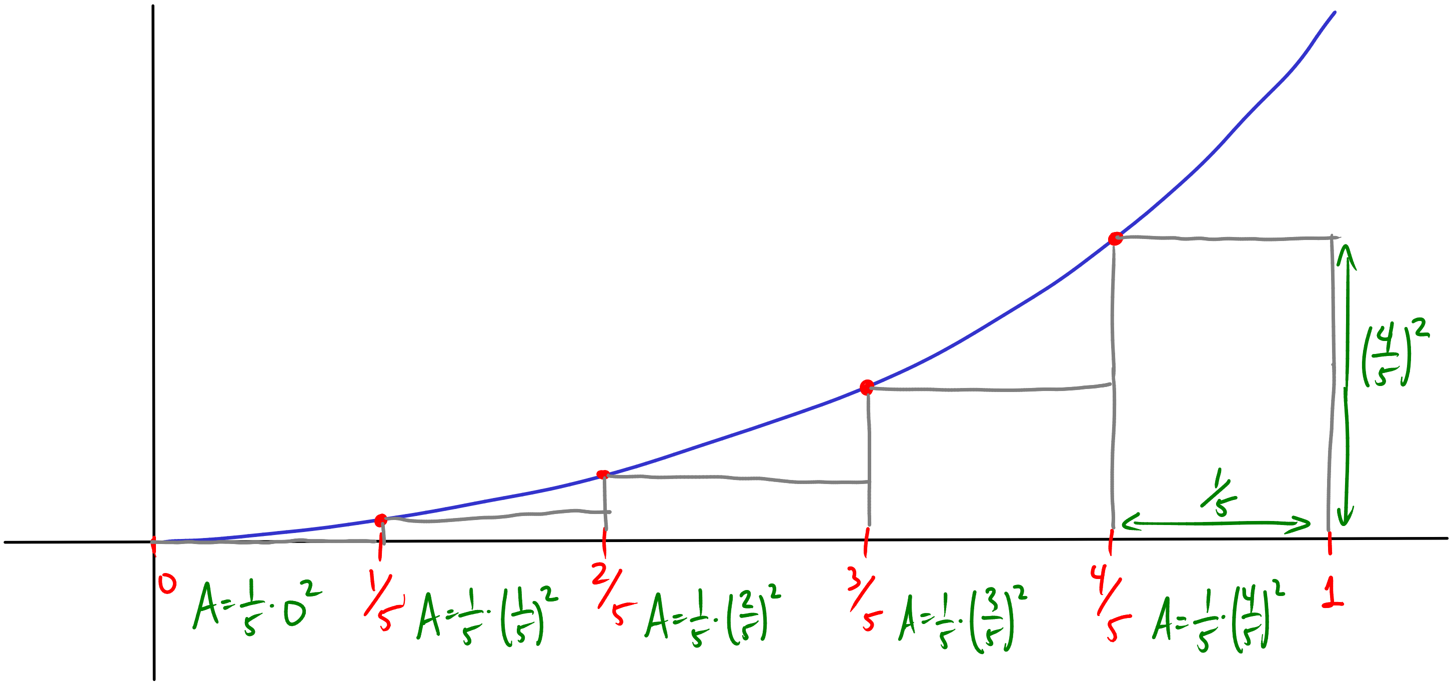
This gives an <u>overestimate</u> of the area under $y = x^2$ from $x = 0$ to $x = 1$.

It is the "estimated area using <u>4 rectangles</u> and using <u>right</u> endpoints of the intervals as sample points."

So call it R_4.

Then $R_4 = \frac{15}{32}$.

Ex Estimate the same area using 5 rectangles and left endpoints as sample points.



Estimated area $$L_5 = \frac{1}{5}\left(0^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2\right)$$

$$= \frac{30}{125}$$

This is an underestimate of the actual area.

Now say we use 100 rectangles.

Then get $$L_{100} = \frac{1}{100}\left(0^2 + \left(\frac{1}{100}\right)^2 + \left(\frac{2}{100}\right)^2 + \cdots + \left(\frac{99}{100}\right)^2\right) = 0.3285$$

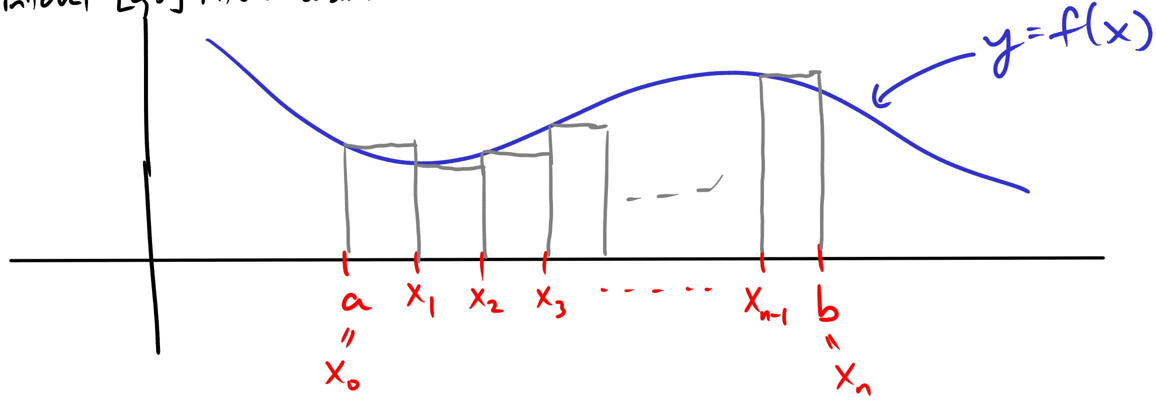
$$R_{100} = \frac{1}{100}\left(\left(\frac{1}{100}\right)^2 + \left(\frac{2}{100}\right)^2 + \left(\frac{3}{100}\right)^2 + \cdots + \left(\frac{100}{100}\right)^2\right) = 0.3385$$

n	L_n	R_n
10	.285	.385
100	.3285	.3385
1000	.33285	.33385

As $n \to \infty$, both L_n and R_n approach $\frac{1}{3}$.

So, $\frac{1}{3}$ is the exact area between the graph $y = x^2$ and the x-axis and between $x = 0$ and $x = 1$.

For any continuous function $f(x)$, estimate the area similarly:
chop interval $[a,b]$ into n subintervals



Width of each rectangle: $\Delta x = \frac{b-a}{n}$

Heights of rectangles: (using left endpoints) $f(x_0), f(x_1), f(x_2), \ldots, f(x_{n-1})$

where $x_0 = a$
$x_1 = a + \Delta x$ i.e. $x_i = a + i \cdot \Delta x$
$x_2 = a + 2\Delta x$
$\vdots$

$\longrightarrow$ estimated area $L_n = \Delta x \cdot \left(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right)$

A convenient notation ("sigma notation"):
the symbol $\sum_{i=1}^{n} a_i$ means $a_1 + a_2 + a_3 + \cdots + a_n$.

Ex What is $\sum_{i=1}^{4} i^2$?

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = \underline{\underline{30}}.$$

Ex What is $\sum_{i=1}^{6} 3i$?

$$3(1)+3(2)+3(3)+3(4)+3(5)+3(6)$$
$$= 3+6+9+12+15+18$$
$$= \underline{\underline{63}}.$$

Ex Write $\frac{1^3}{n} + \frac{2^3}{n} + \frac{3^3}{n} + \cdots + \frac{n^3}{n}$ in sigma notation.

$$\sum_{i=1}^{n} \frac{i^3}{n}.$$

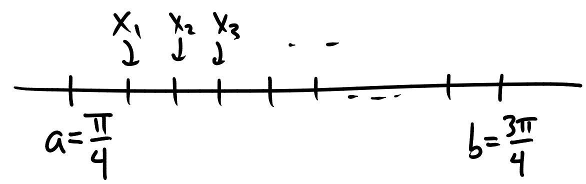
In this notation, $L_n = \Delta x \sum_{i=1}^{n} f(x_{i-1})$.

$$R_n = \Delta x \cdot \sum_{i=1}^{n} f(x_i)$$

The actual area is $A = \lim_{n\to\infty} L_n$. or, $A = \lim_{n\to\infty} R_n$.

(Both are the same!)

Ex Let A be the area of the region under the graph of $f(x) = \sin^2 x$ between $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$. Using right endpoints as sample points,



- Write a formula for A as a limit,

$a = \frac{\pi}{4}$

$b = \frac{3\pi}{4}$

$$\Delta x = \frac{b-a}{n} = \frac{\frac{3\pi}{4} - \frac{\pi}{4}}{n} = \frac{\pi}{2n}$$

$$x_i = a + i\Delta x = \frac{\pi}{4} + i \cdot \frac{\pi}{2n}$$

$$R_n = \Delta x \cdot \sum_{i=1}^{n} f(x_i) = \frac{\pi}{2n} \cdot \sum_{i=1}^{n} \sin^2(x_i)$$

$$= \frac{\pi}{2n} \cdot \sum_{i=1}^{n} \sin^2\left(\frac{\pi}{4} + i \cdot \frac{\pi}{2n}\right)$$

The actual area is $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{\pi}{2n} \sum_{i=1}^{n} \sin^2\left(\frac{\pi}{4} + i \cdot \frac{\pi}{2n}\right)$

- Estimate A using 3 rectangles.

$$R_3 = \frac{\pi}{6} \cdot \sum_{i=1}^{3} \sin^2\left(\frac{\pi}{4} + i \cdot \frac{\pi}{6}\right)$$

$$= \frac{\pi}{6}\left(\sin^2\left(\frac{\pi}{4} + \frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{4} + 2 \cdot \frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{4} + 3 \cdot \frac{\pi}{6}\right)\right)$$

$$\approx 1.23885$$

Definite integrals

Say $f(x)$ is a function defined for $a \le x \le b$.

Divide $[a,b]$ into n equal subintervals of width Δx, endpoints $\underset{\overset{\shortparallel}{a}}{x_0}, x_1, x_2, \ldots, \underset{\overset{\shortparallel}{b}}{x_n}$

$(x_i = a + i\Delta x)$

Pick any "sample points" x_i^* in $[x_{i-1}, x_i]$.

The definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\left(\sum_{i=1}^{n} f(x_i^*)\cdot \Delta x\right)}_{\text{"Riemann sum"}}$$

if that limit exists!

(It always exists, if f is continuous.)