Lecture 20

10 Nov 2015

Last time: integrals and calculating them using FTC.

FTC:

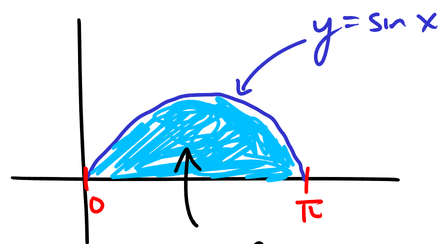
I. $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$

i.e. $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$.

II. $\int_a^b f(x)\,dx = F(b) - F(a) = F\Big|_a^b$ where F is any antiderivative of f.

Ex $\frac{d}{dx}\left[\int_3^x \cos(t^7)\,dt\right] = \cos(x^7)$ by FTC I

Ex $\int_0^\pi \sin x\,dx = 2$ (last time) by FTC II

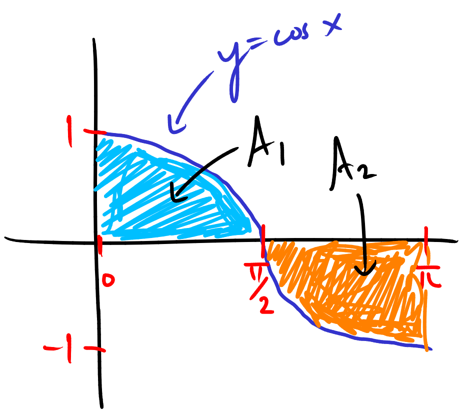


Ex $\int_0^\pi \cos x\,dx = ?$ and what does it mean in terms of areas?

an antideriv. of $\cos x$ is $\sin x$, so $\int_0^\pi \cos x\,dx = \sin x\Big|_0^\pi$

$= \sin\pi - \sin 0$

$= 0 - 0 = 0$.



$\int_0^\pi \cos x\,dx = A_1 - A_2 = 0$ ($A_1 = A_2$ by symmetry)

Ex Calculate $\int_{\pi/4}^{\pi/3} \sec\theta \tan\theta \, d\theta$.

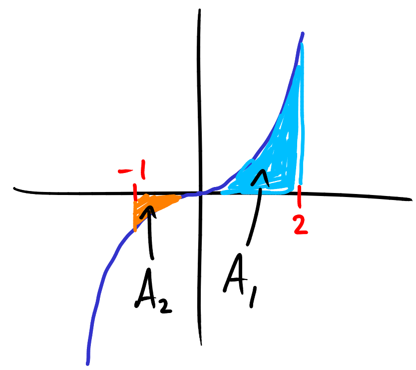
Use FTC II. Remember: $\frac{d}{d\theta}(\sec\theta) = \sec\theta\tan\theta$.

So, $\sec\theta$ is an antiderivative of $\sec\theta\tan\theta$.

So, $\int_{\pi/4}^{\pi/3} \sec\theta\tan\theta \, d\theta = \sec\theta \Big|_{\pi/4}^{\pi/3} = \sec\left(\frac{\pi}{3}\right) - \sec\left(\frac{\pi}{4}\right)$

$$= \underline{\underline{2 - \sqrt{2}}}$$

Ex Calculate $\int_{-1}^{2} x^3 \, dx$ and interpret it as a difference of areas.



$$\int_{-1}^{2} x^3 \, dx = \frac{1}{4}x^4 \Big|_{-1}^{2} = \frac{1}{4}(2)^4 - \frac{1}{4}(-1)^4$$

$$= \frac{1}{4}(16) - \frac{1}{4}(1) = \frac{1}{4}(15) = \underline{\underline{\frac{15}{4}}}.$$

So $\frac{15}{4} = A_1 - A_2$.

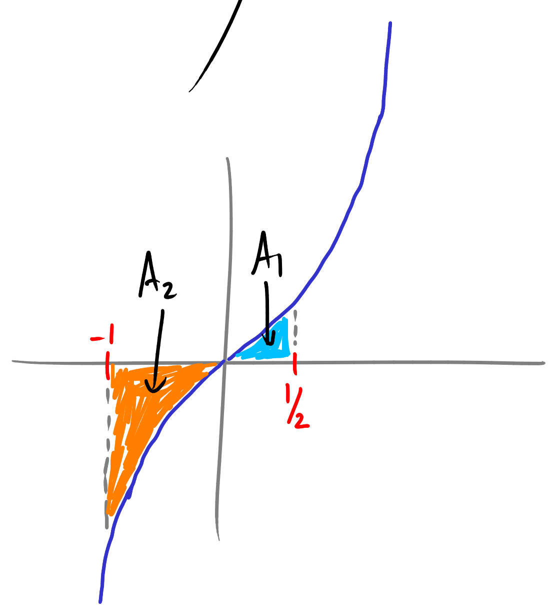
(Remark: if we use another antiderivative we still get the same answer, e.g.

$$\int_{-1}^{2} x^3 \, dx = \frac{1}{4}x^4 + 1 \Big|_{-1}^{2} = \left(\frac{1}{4}(2^4) + 1\right) - \left(\frac{1}{4}(-1)^4 + 1\right)$$

$$= \frac{1}{4}(16) \cancel{+1} - \frac{1}{4}(1) \cancel{-1}$$

$$= \frac{1}{4}(16-1) = \underline{\underline{\frac{15}{4}}}.$$
)

Ex Is $\int_{-1}^{1/2} \tan x \, dx$ positive, negative or zero?



$\int_{-1}^{1/2} \tan x \, dx = A_1 - A_2$

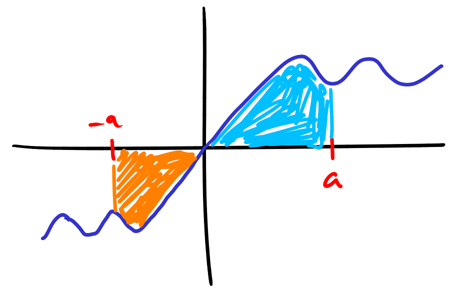
and $A_2 > A_1$, so $\int_{-1}^{1/2} \tan x \, dx < 0$.

Ex Is $\int_{-1/2}^{1/2} \tan x \, dx$ positive, negative or zero? It's zero, +ve and -ve areas cancel each other.

General rule: integrals of symmetric functions

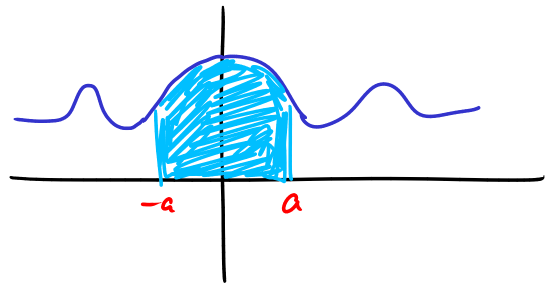
a) If f is odd, $f(-x) = -f(x)$

then $\int_{-a}^{a} f(x)\,dx = 0$.



b) If f is even, $f(x) = -f(x)$

then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



Ex $$\int_{-0.154}^{0.154} \frac{\overbrace{(\tan x)}^{\text{odd}}\overbrace{\left(x^6+29x^4+\frac{105}{4}x^2+980.7\right)}^{\text{even}}}{\underbrace{x^{12}+7\cos(32x)}_{\text{even}}}\,dx = 0$$

by rule a).

Ex $$\int_{-2}^{2} x^4+8x^3+9x \, dx = \int_{-2}^{2} x^4 \, dx + \int_{-2}^{2} 8x^3+9x \, dx$$

$$= \int_{-2}^{2} x^4 \, dx + 0 \qquad \text{by rule a)}$$

$$= 2\int_{0}^{2} x^4 \, dx \qquad \text{by rule b)}$$

$$= 2\left.\frac{x^5}{5}\right|_0^2 = 2\cdot\frac{32}{5} = \underline{\underline{\frac{64}{5}}}$$

Ex $$\int_{\pi/6}^{\pi/3} \left(-\frac{3}{\sin^2\theta}+\theta\right) d\theta$$

$$= \int_{\pi/6}^{\pi/3} -3\csc^2\theta + \theta \, d\theta$$

$$= 3\cot\theta + \frac{\theta^2}{2} \Big|_{\pi/6}^{\pi/3}$$

$$= \cdots = \underline{\underline{-2\sqrt{3} + \frac{\pi^2}{24}}}$$

Indefinite integrals

Notation: $\int f(x)\,dx$ (with no limits a,b) means <u>any antiderivative</u> of $f(x)$.

Ex $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

$\int \sin x\,dx = -\cos x + C$ $\qquad$ $\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$ $\qquad$ $\int \csc^2 x\,dx = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$ $\qquad$ $\int \csc x \cot x\,dx = -\csc x + C$

Ex Find $\int (10x^4 + 6\sec^2 x)\,dx$.

$$= 10\left(\frac{x^5}{5}\right) + 6\tan x + C = \underline{\underline{2x^5 + 6\tan x + C}}$$

Ex Find $\int_0^{\pi/4} (10x^4 + 6\sec^2 x)\,dx$.

$$= 2x^5 + 6\tan x + C \Big|_0^{\pi/4} = \left(2\left(\frac{\pi}{4}\right)^5 + 6\cdot\tan\left(\frac{\pi}{4}\right) + C\right) - \left(2\cdot 0^5 + 6\tan 0 + C\right)$$

$$= \left(\frac{\pi^5}{512} + 6 + C\right) - (0 + 0 + C)$$

$$= \underline{\underline{\frac{\pi^5}{512} + 6}}$$

Ex Find $\int u^{2/3}\,du$. $\qquad u^n \quad n=\frac{2}{3} \quad n+1=\frac{5}{3}$

$$= \frac{u^{5/3}}{\left(\frac{5}{3}\right)} + C = \frac{3}{5}u^{5/3} + C$$

Ex Find $\int_1^8 u^{2/3}\,du$.

$$= \frac{3}{5}u^{5/3}\Big|_1^8 = \frac{3}{5}\left(8^{5/3}\right) - \frac{3}{5}\left(1^{5/3}\right)$$

$$= \frac{3}{5}\left(8^{5/3} - 1^{5/3}\right)$$

$$= \frac{3}{5}(32-1)$$

$$= \underline{\underline{\frac{93}{5}}}$$

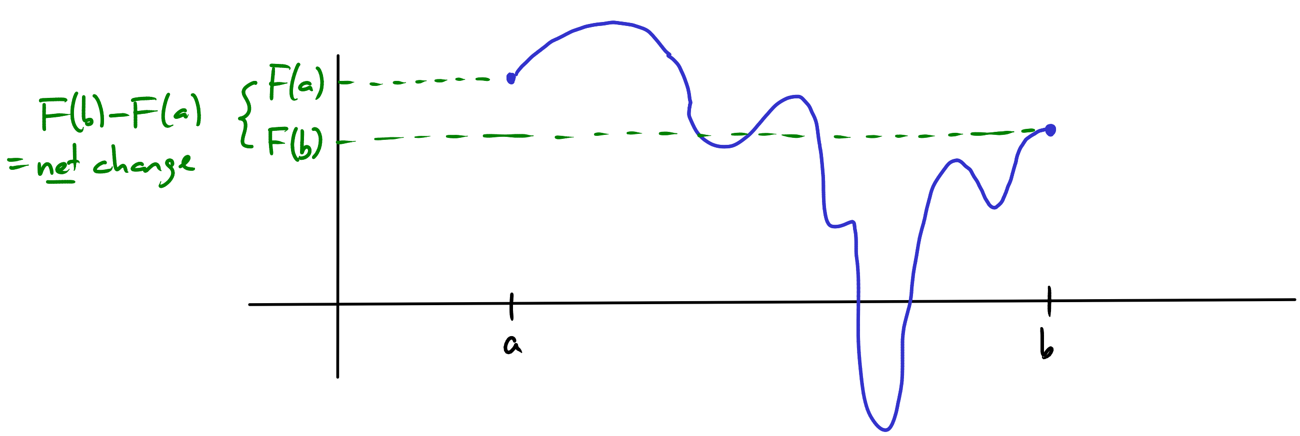
Another use/interpretation of FTC II: <u>Net Change Theorem</u>

Given a function $F(t)$ $\qquad t = \text{time}$

$F'(t)$ is the <u>rate of change</u> of $F(t)$.

FTC II says:

$$\int_a^b F'(t)\,dt = F(b) - F(a) = \text{net change of } F \text{ over the time interval } [a,b].$$



Ex Water flows into a reservoir at the rate $(10t+6)$ ft^3/s. (t in sec)

The reservoir contains 400 ft^3 of water at time $t=0$.

How much does it contain at $t=10$s?

The net change from $t=0$ to $t=10$ is

$$\int_0^{10} (10t+6)\,dt = 5t^2+6t\Big|_0^{10}$$
$$= \left(5(10)^2+6(10)\right) - \left(5(0)^2+6(0)\right)$$
$$= (500+60) - (0+0) = \underline{560}. \qquad (560 \text{ ft}^3)$$

So at $t=10$s we have $400+560 = \underline{\underline{960 \text{ ft}^3}}$ of water.

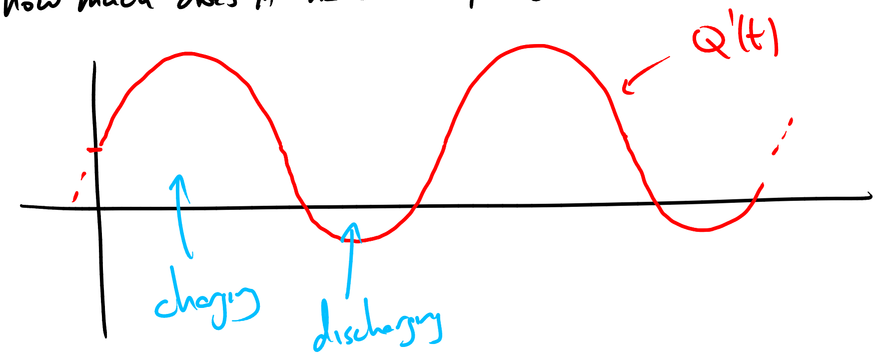
Ex A capacitor is connected to a load that can charge or discharge it.

The current flowing into the capacitor is $Q'(t) = \frac{1}{2} + \sin(\pi t)$.

($Q(t)$ = charge of cap. at time t)

If the cap. starts with 10 units of charge at $t=0$ $(Q(0)=10)$

how much does it have at $t=6$?



$$Q(6)-Q(0) = \int_0^6 Q'(t)\,dt = \int_0^6 \left(\sin\pi t + \tfrac{1}{2}\right) dt$$
$$= -\frac{1}{\pi}\cos\pi t + \frac{t}{2}\Big|_0^6$$
$$= \left(-\frac{1}{\pi}\cos 6\pi + 3\right) - \left(-\frac{1}{\pi}\cos 0 + 0\right)$$

$$= \left(-\frac{1}{\pi}+3\right) - \left(-\frac{1}{\pi}\right) = 3.$$

So $Q(b) = Q(0) + 3$

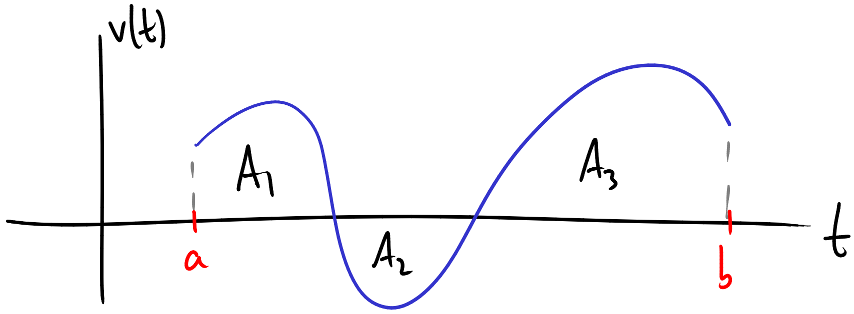
$= 10 + 3 = \underline{\underline{13}}$ units.

A standard example of net change: <u>total displacement</u>.

If $s(t)$ = position along a line

$s'(t) = v(t)$ = velocity

$\left(\begin{array}{ll} v(t)>0: & \text{moving to the right} \\ v(t)<0: & \text{" " " left} \end{array}\right)$



<u>Total displacement</u> $\quad s(b)-s(a) = \int_a^b v(t)\,dt = A_1 - A_2 + A_3$

<u>Total distance</u> $\quad \int_a^b |v(t)|\,dt = A_1 + A_2 + A_3.$

(odometer reading)