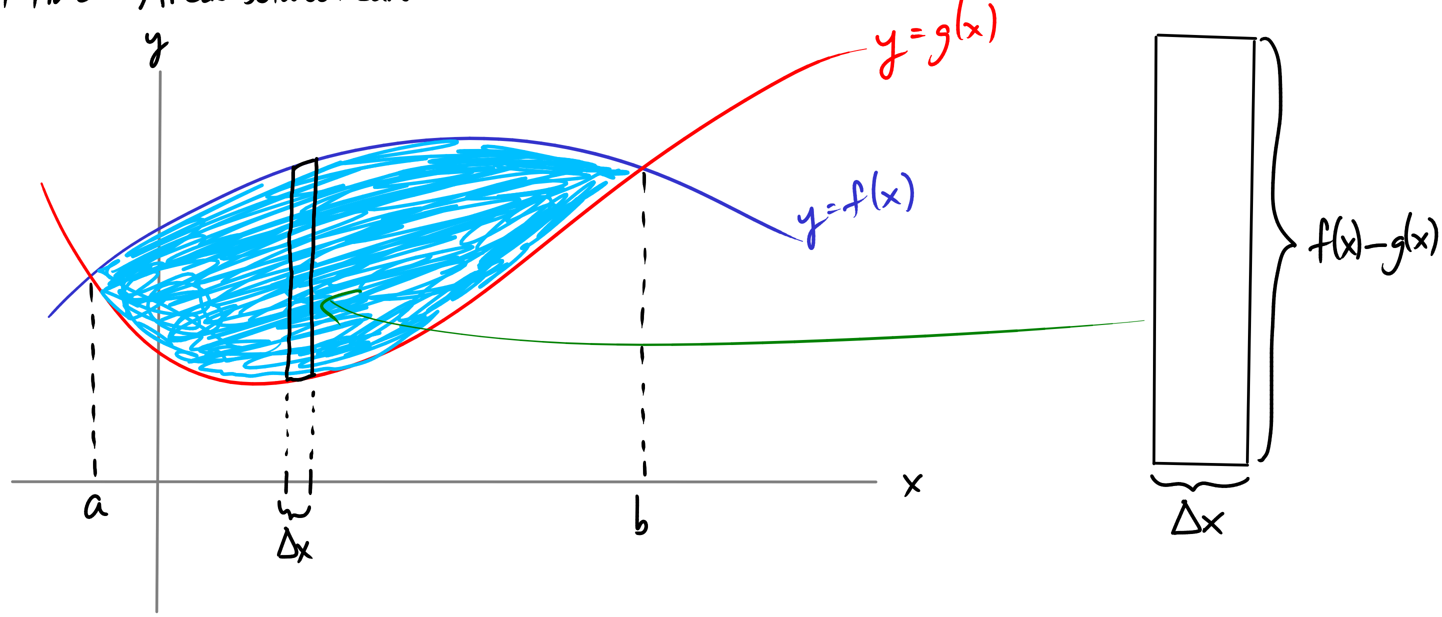
Lecture 23

19 Nov 2015

Last time: Areas between curves



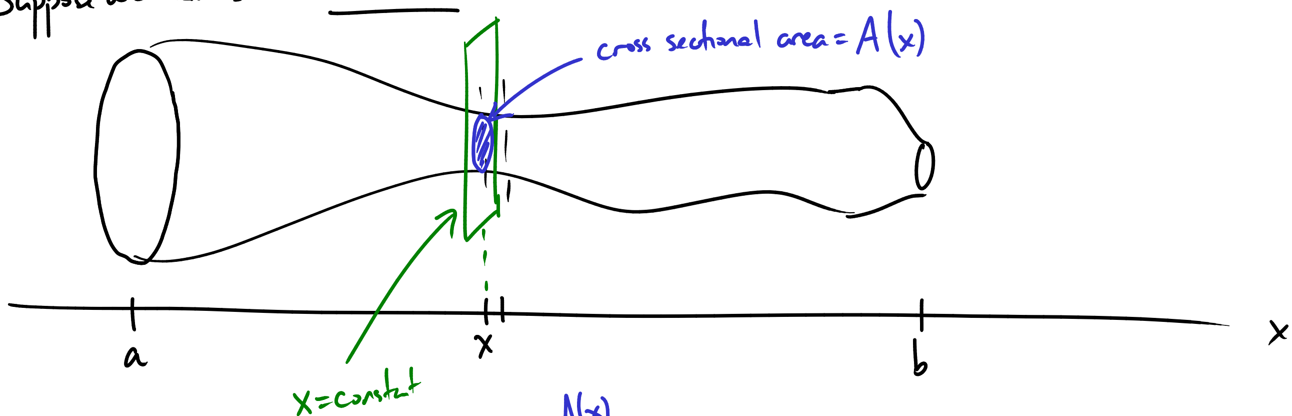
rectangle has area $= \Delta x\big(f(x)-g(x)\big)$

Sum all the rectangles, take limit $\Delta x \to 0$

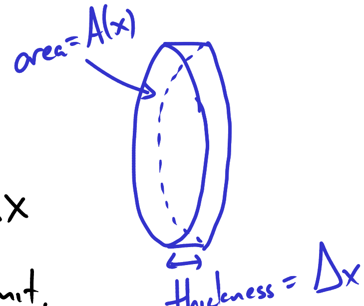
get total area: $A = \int_a^b dx\,\big(f(x)-g(x)\big)$

Volumes (Ch 6.2)

Suppose we have some 3-d solid and we want to calculate its volume.



We chop the solid into slices:

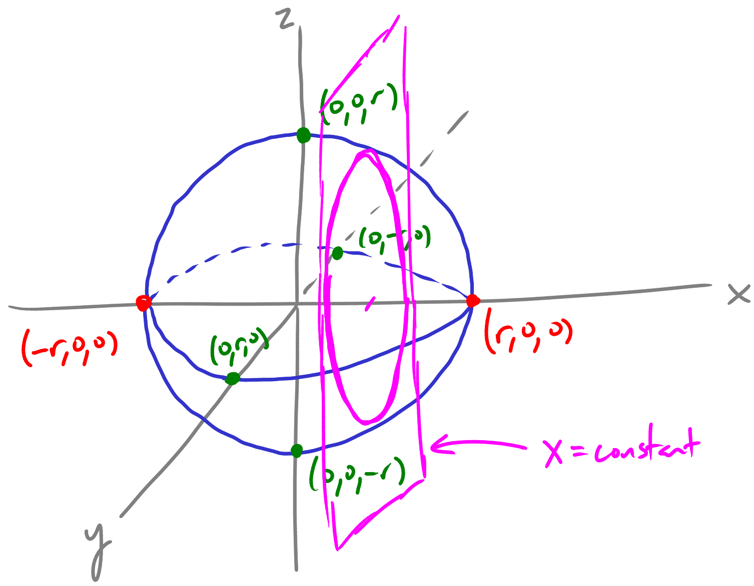


volume of the slice $= A(x)\cdot\Delta x$

Add them all up, take $\Delta x \to 0$ limit.

$\to$ Total volume of the solid: $V = \int_a^b A(x)\,dx$

Ex Calculate the volume of a ball of radius r.



The ball is

$$x^2+y^2+z^2 \leq r^2$$

Fixing a plane at constant x:

$$y^2+z^2 \leq r^2-x^2$$

i.e. (y,z) lie inside circle of radius $\sqrt{r^2-x^2}$

So, the cross sections are circles, with area $A(x) = \pi\left(\sqrt{r^2-x^2}\right)^2$

i.e. $A(x) = \pi(r^2-x^2)$

Volume of sphere: $V = \int_{-r}^{r} A(x)\,dx$

$$= \int_{-r}^{r} \pi(r^2-x^2)\,dx$$

$$= \pi\left(r^2x - \frac{1}{3}x^3 \Big|_{-r}^{r}\right)$$

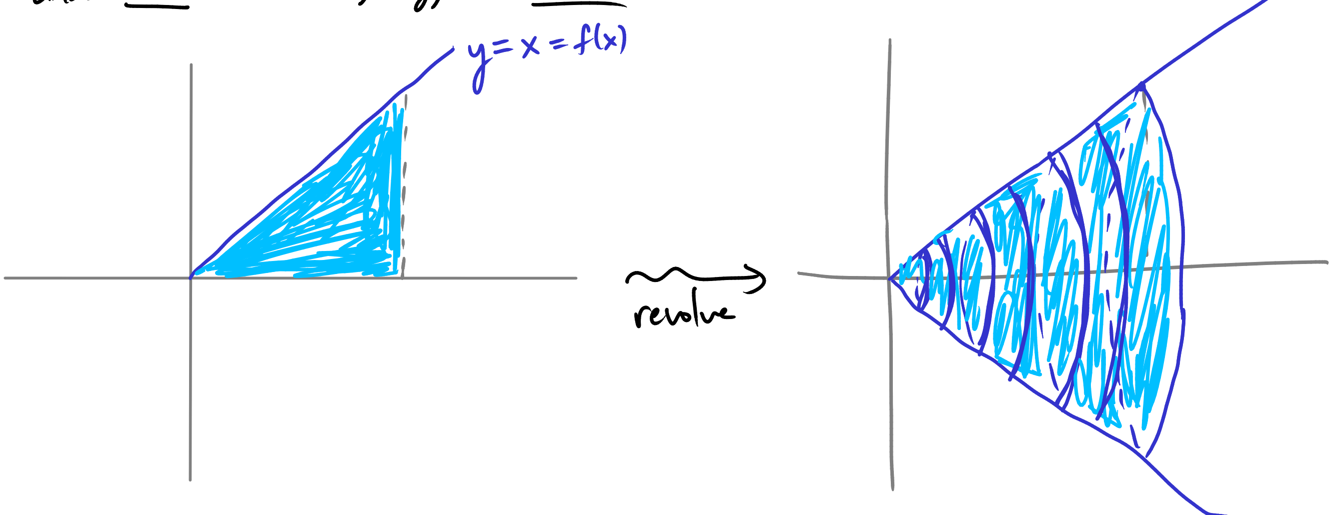
$$= \pi\left(\left(r^3-\frac{1}{3}r^3\right) - \left(-r^3+\frac{1}{3}r^3\right)\right)$$

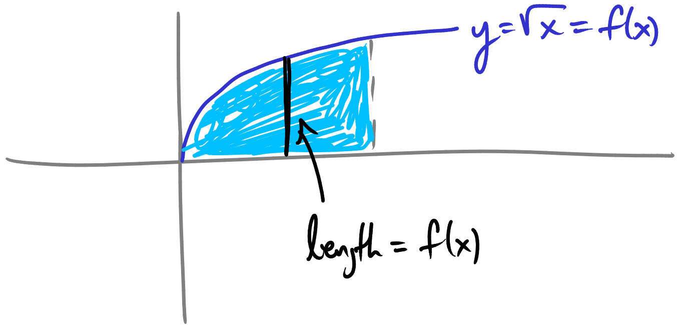
$$= \pi\left(\frac{2}{3}r^3 - \left(-\frac{2}{3}r^3\right)\right)$$

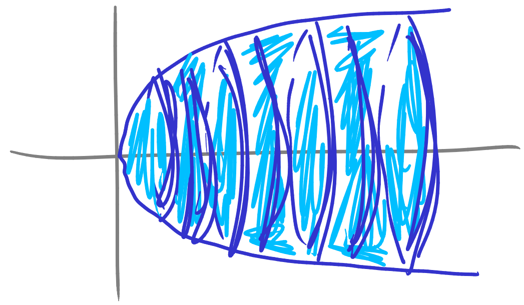
$$= \underline{\pi \cdot \frac{4}{3}r^3}$$

A common type of solid: "solid of revolution" — take some region of the plane and revolve it around, say, the x-axis.

Ex







The cross-section of either of these solids at fixed x, is a circle of radius $f(x)$.

So the cross-section area is $A(x)=\pi\cdot f(x)^2$.

Ex Find the volume of a solid obtained by revolving the region between $y=\sqrt{x}$, $y=0$, $x=2$ around the x-axis.

$$V=\int_0^2 A(x)\,dx = \int_0^2 \pi\cdot(\sqrt{x})^2\,dx = \int_0^2 \pi x\,dx = \pi\frac{x^2}{2}\Big|_0^2 = \pi(2-0) = 2\pi$$

Can also revolve around e.g. the y-axis.

Ex Find the volume of a solid obtained by revolving the region between

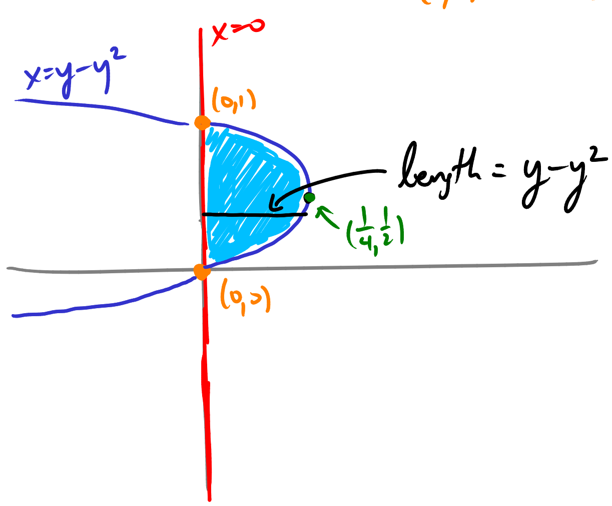
$x=y-y^2$

$x=0$

around the y-axis.

Intersections: $\left.\begin{array}{l} x=0 \\ x=y-y^2 \end{array}\right\} \to y-y^2=0$, i.e. $y(1-y)=0$, i.e. $y=0$ or $y=1$

$(0,0)$ $(0,1)$



Slice horizontally: i.e. look at cross-sections at fixed value of y.

The cross-sections are circles, with radius depending on y $(f(y))$

radius $= y-y^2$

area $= A(y)=\pi(y-y^2)^2$

$$\text{total volume} = \int_0^1 dy\, A(y) = \int_0^1 dy\, \pi \cdot (y-y^2)^2$$
$$= \pi \int_0^1 y^2 - 2y^3 + y^4 \, dy$$
$$= \dots = \frac{\pi}{30}$$

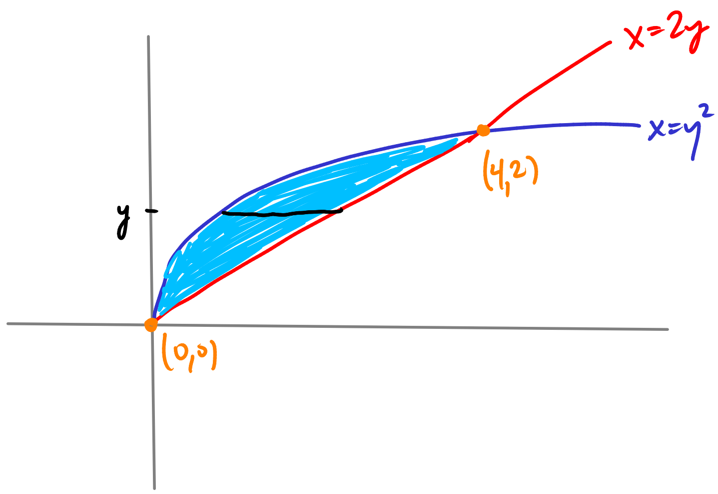
We may also get cross-sections which are "washers"

Ex Let R be the region between $y=\sqrt{x}$ and $x=2y$.
Find the volume of the solid obtained by rotating R around the y-axis.

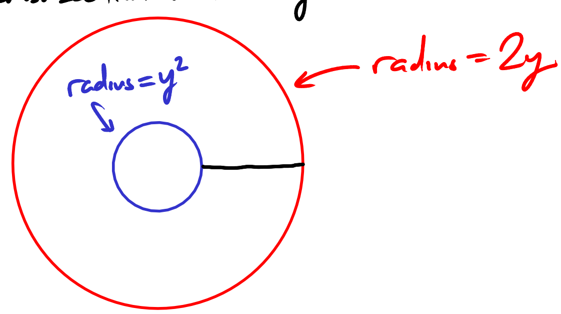
Intersections: $y=\sqrt{x} \longrightarrow y^2=x$, $2y=x$ $\longrightarrow$ $2y=y^2$, i.e. $y^2-2y=0$, i.e. $y(y-2)=0$

so $y=0$ or $y=2$

$(0,0)$ $(4,2)$



Rotate around y-axis.
Cross section at fixed y:



$$A(y) = \pi(2y)^2 - \pi(y^2)^2$$

add up all the slices: $V = \int_0^2 A(y)\, dy$

$$= \pi \int_0^2 4y^2 - y^4 \, dy = \dots = \frac{64}{15}\pi$$