Lecture 25

3 Dec 2015

Midterm grades + solutions posted tomorrow night   average $\approx 82\%$

Final exam: Sat Dec 12   9–11:30   CLA 0.126

comprehensive (incl. volumes, average values)

~ 28 questions

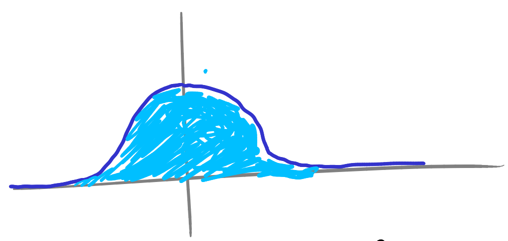
Extra office hrs:

Friday 2:30

Tue 2:30

Something fun:

Recall we can't write an antiderivative for $f(x) = e^{-x^2}$ (in terms of elementary functions)

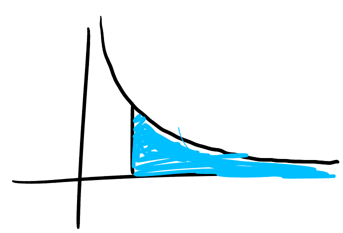


But, amazingly:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

① What does this mean?

$$\int_c^{\infty} f(x)\,dx = \lim_{L\to\infty}\int_c^L f(x)\,dx$$



e.g.

$$\int_1^{\infty}\frac{1}{x^2}\,dx = \lim_{L\to\infty}\int_1^L \frac{1}{x^2}\,dx$$

$$= \lim_{L\to\infty}\left(-\frac{1}{x}\Big|_1^L\right)$$

$$= \lim_{L\to\infty}\left(1-\frac{1}{L}\right)$$

$$= 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{M\to-\infty}\lim_{L\to\infty}\int_M^L f(x)\,dx$$

② How do we calculate it?

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

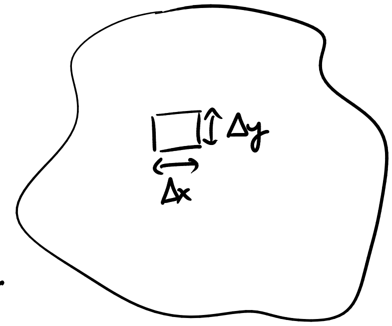
Trick:

$$I \cdot I = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$$

ie $$I^2 = \iint_{-\infty\,-\infty}^{\infty\;\;\infty} e^{-x^2-y^2}\,dx\,dy$$

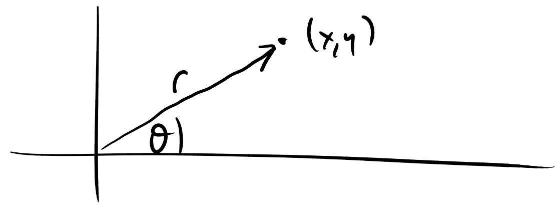
"double integral"

(sum up $f(x,y)\,\Delta x\,\Delta y$ and take limit as $\Delta x, \Delta y \to 0$)

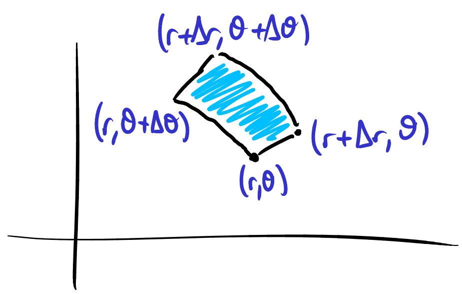


Switch to polar coordinates: (r, θ)

$$x = r\cos\theta$$
$$y = r\sin\theta$$



Instead of chopping plane into little rectangles, chop it into angular sectors



area $\approx r\,\Delta r\,\Delta\theta$

$$I^2 = \iint e^{-x^2-y^2}\,dx\,dy$$

$x^2+y^2 = r^2$

$dx\,dy \longrightarrow r\,dr\,d\theta$

$$= \iint e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\int_0^{\infty} e^{-r^2}\,r\,dr\right]d\theta$$

$$= \int_0^{2\pi}\left(-\frac{1}{2}e^{-r^2}\Big|_0^{\infty}\right)d\theta$$

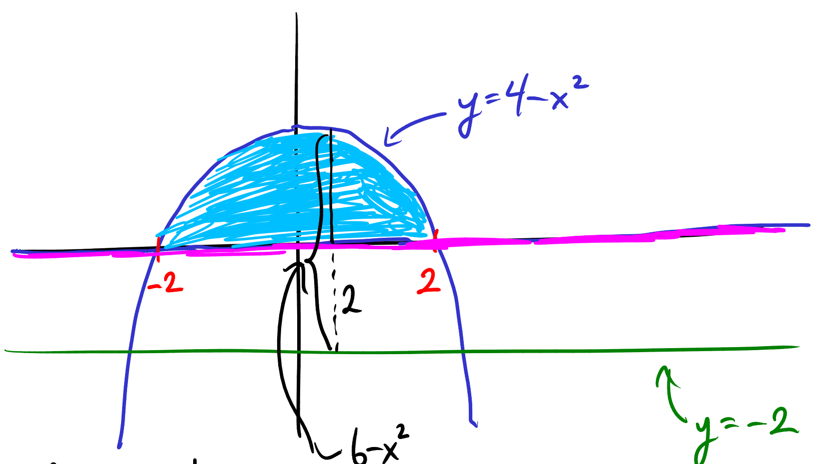
$$= \int_0^{2\pi} \left(\lim_{L\to\infty} \left(-\tfrac{1}{2}e^{-L^2} + \tfrac{1}{2}\right)\right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\, d\theta = \frac{1}{2}(2\pi) = \underline{\underline{\pi}}$$

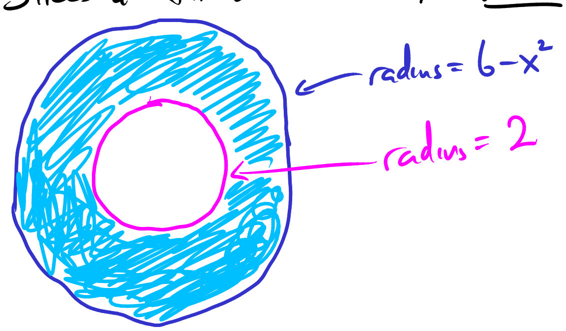
so $I^2 = \pi$, so $\underline{\underline{I = \sqrt{\pi}}}$.

Volumes of revolution around other axes

Ex Let R be the region bounded by $y=0$ and $y=4-x^2$. Find the volume of the solid obtained by revolving R around the line $y=-2$.



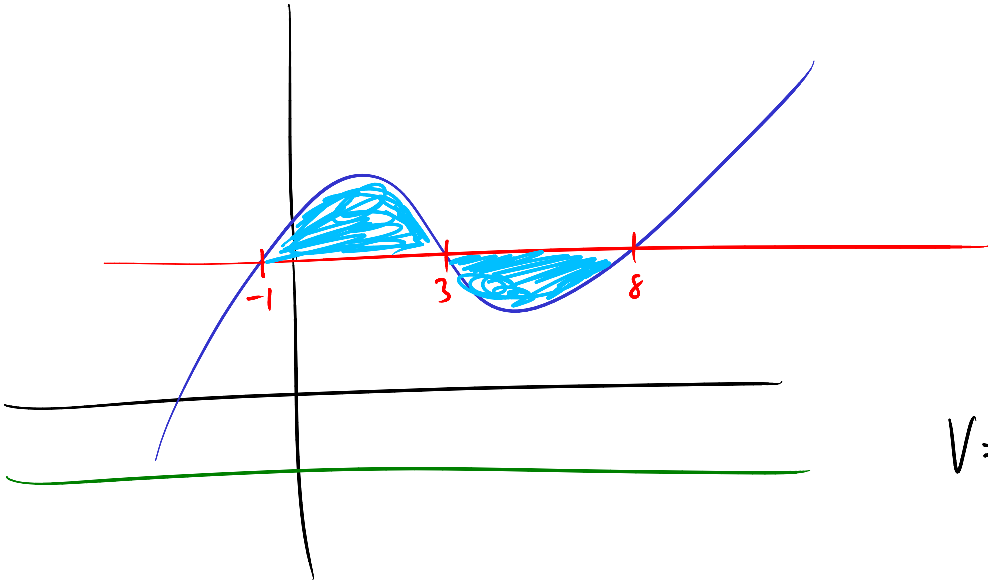
Slices at fixed value of x: washers



x

$$A(x) = \pi\left(6-x^2\right)^2 - \pi(2)^2$$
$$= \pi\left(36 - 12x^2 + x^4 - 4\right)$$
$$= \pi\left(x^4 - 12x^2 + 32\right)$$

$$V = \int_{-2}^{2} A(x)\, dx$$
$$= \int_{-2}^{2} \pi\left(x^4 - 12x^2 + 32\right) dx = \cdots$$

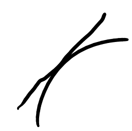


$$V = \int_{-1}^{3} A(x)\,dx + \int_{3}^{8} A(x)\,dx$$

We know that for a smooth function $f(x)$

have <u>linear approximation</u>

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$



To get <u>better</u> approx:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{6}(x - x_0)^3$$

... and eventually,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

e.g.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$