Lecture 25

3 Dec 2015

Midterm 3 grades + solutions posted tomorrow night average ≈ 82%

Final exam: Wed Dec 9 2–4:30 pm RLM 4.102

comprehensive (incl. volumes, average values)

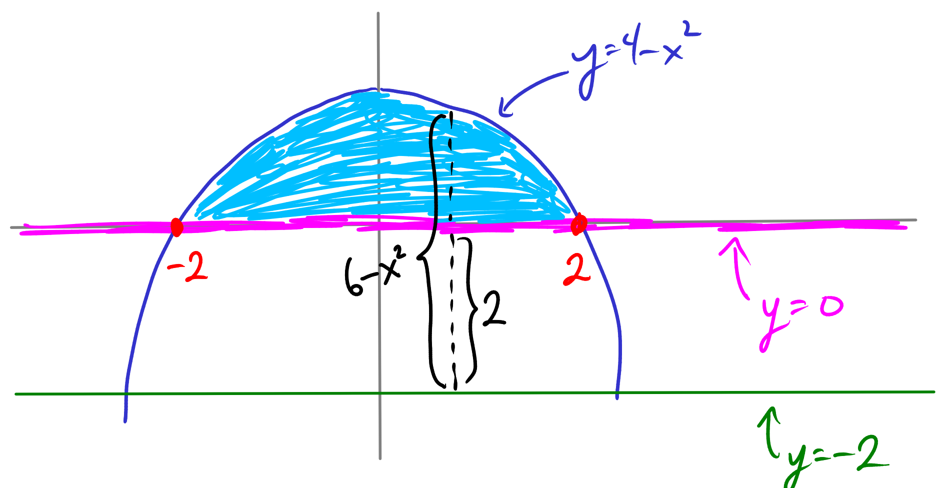
~ 28 questions

Office hrs Tue 2:30

Fri 2:30

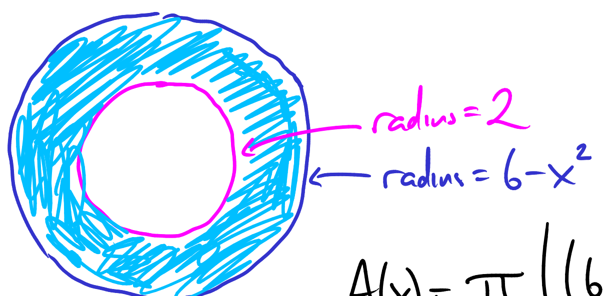
Volumes of revolution around other lines

Ex Let R be the region bounded by $y=0$ and $y=4-x^2$.
Find the volume of the solid obtained by revolving R around the line $y=-2$.



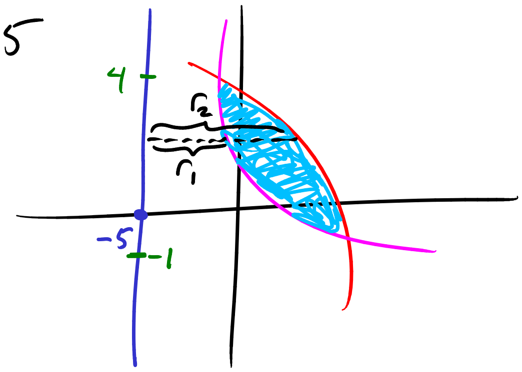
Want to find $A(x)$, the cross section area of slice at fixed x.

Cross section:



$$A(x) = \pi\left((6-x^2)^2\right) - \pi\left(2^2\right)$$
$$= \pi\left(36-12x^2+x^4-4\right)$$
$$= \pi\left(x^4-12x^2-32\right)$$
$$V = \int_{-2}^{2} A(x)\,dx = \int_{-2}^{2} \pi\left(x^4-12x^2-32\right)dx$$
$$= \cdots$$

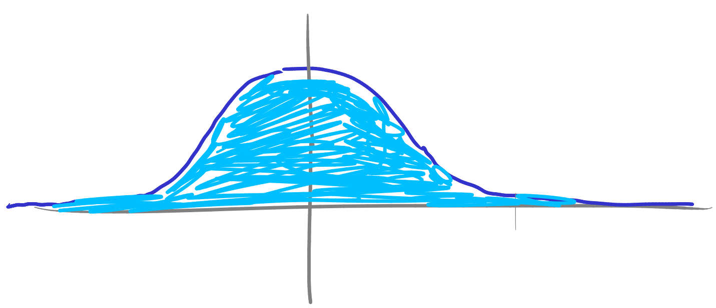
Revolving around the line $x+5=0$: $x=-5$

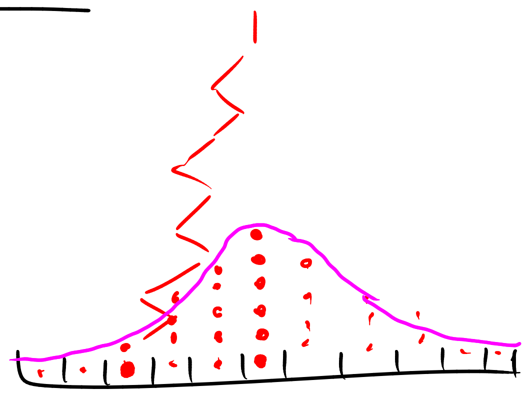


$$A(y) = \pi\left(r_2^2 - r_1^2\right)$$
$$V = \int_{-1}^{4} A(y)\, dy$$

Something fun:

Recall we can't write a formula for antiderivative of $f(x) = e^{-x^2}$





But amazingly: $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$

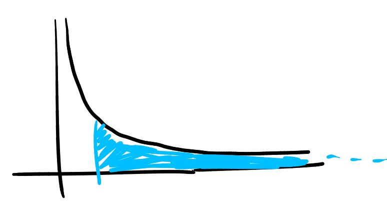
① What does this mean?

$$\int_c^{\infty} f(x)\, dx = \lim_{L\to\infty} \int_c^L f(x)\, dx$$

e.g. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{L\to\infty} \int_1^L \frac{1}{x^2}\, dx$

$$= \lim_{L\to\infty} \left(-\frac{1}{x}\Big|_1^L\right)$$

$$= \lim_{L\to\infty} \left(1 - \frac{1}{L}\right) = \underline{1}$$



$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{M\to-\infty} \lim_{L\to\infty} \int_M^L f(x)\,dx$$

② How do we calculate it?

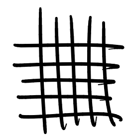
$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

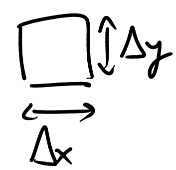
Trick: $I \cdot I = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

"double integral"

chop up x-y plane into little rectangles





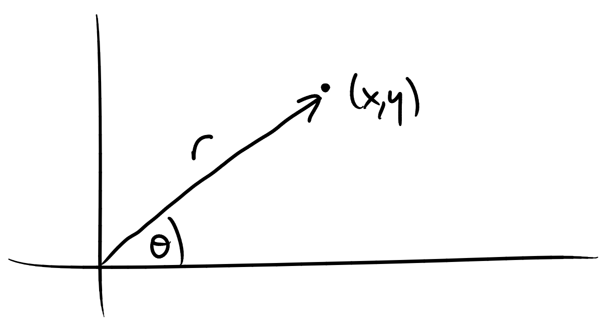
area $= \Delta x \Delta y \longrightarrow dx\,dy$

sample $f(x,y) = e^{-x^2-y^2}$ in each rectangle, take "Rieman sum"

$$\sum_i f(x_i^*, y_i^*)\,\Delta x \Delta y$$

Trick: switch to polar coordinates

(r,θ)



$x = r\cos\theta$

$y = r\sin\theta$

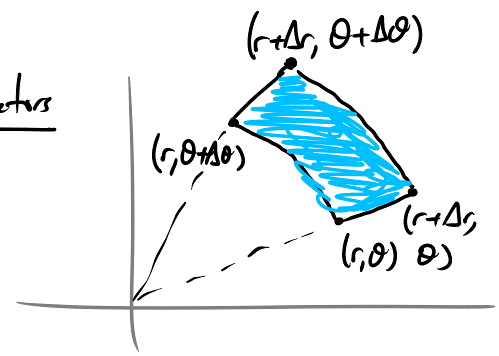
$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$= r^2$

so $e^{-x^2-y^2} = e^{-r^2}$

Instead of chopping plane into little rectangles, chop it into angular sectors

Area of sector $\approx r\,\Delta r\,\Delta\theta$



$$I^2 = \iint e^{-x^2-y^2}\, dx\, dy$$

$$x^2+y^2=r^2$$
$$dx\, dy \longrightarrow r\, dr\, d\theta$$

$$= \iint e^{-r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\infty} e^{-r^2}\, r\, dr \right] d\theta$$

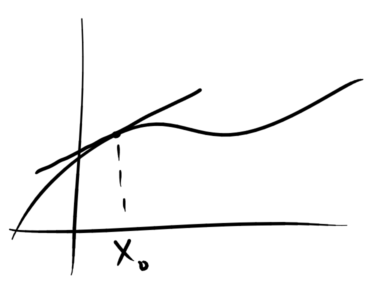
$$= \int_0^{2\pi} \left(-\frac{1}{2} e^{-r^2} \Big|_0^{\infty} \right) d\theta$$

$$= \int_0^{2\pi} \left(\lim_{L\to\infty} -\frac{1}{2} e^{-L^2} + \frac{1}{2} \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\, d\theta = \frac{1}{2}(2\pi) = \pi$$

So, $\underline{\underline{I = \sqrt{\pi}}}$

Remember $\underline{\text{linear approximation}}$ $\quad f(x) \approx f(x_0) + f'(x_0)(x-x_0)$



Can do better: $\quad f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \dfrac{f''(x_0)}{2}(x-x_0)^2$

even better: $\quad f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \dfrac{f''(x_0)}{2}(x-x_0)^2 + \dfrac{f'''(x_0)}{6}(x-x_0)^3$

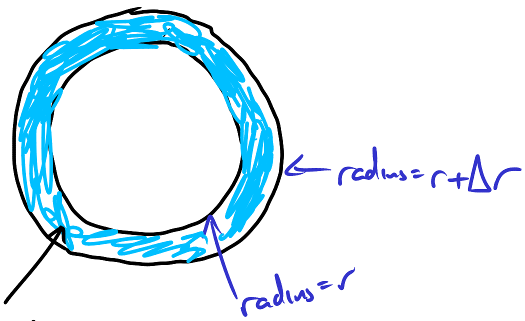
...

and eventually, $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

e.g. $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Recall: $A(r) = \pi r^2$ <u>area of circle</u>

$$\frac{dA}{dr} = 2\pi r = \text{circumference of circle}$$ <u>why?</u>



area of washer

$\approx$ (circumference) $\cdot \Delta r$

$$\frac{dA}{dr} \approx \frac{(\text{area of bigger circle}) - (\text{area of smaller circle})}{\Delta r}$$

$$\approx \frac{(\text{area of washer})}{\Delta r}$$

$\approx$ <u>circumference</u>!