Lecture 3

5 Sep 2018

HW01 was due Tue 3am

HW02 is due next Tue 3am

⋮

Last time: composition of functions

$$(f \circ g)(x) = f(g(x))$$

$$x \to \underbrace{\boxed{g} \to g(x) \to \boxed{f}}_{f \circ g} \to f(g(x))$$

Inverse functions

If f is a function, its inverse is a function f^{-1} such that

$$(f^{-1} \circ f)(x) = x$$

$\text{domain}(f^{-1}) = \text{range}(f)$

$\text{range}(f^{-1}) = \text{domain}(f)$

$$x \to \boxed{f} \to f(x) \to \boxed{f^{-1}} \to x$$

x	$f(x)$
1	7
2	13
3	100

x	$f^{-1}(x)$
7	1
13	2
100	3

Ex If $f(x) = 13x$ then $f^{-1}(x) = \dfrac{x}{13}$

(because $f^{-1}(f(x)) = f^{-1}(13x) = \dfrac{13x}{13} = x$)

Ex If $f(x) = x+3$ then $f^{-1}(x) = x-3$

Some functions f don't have inverses. eg

x	$f(x)$
1	8
2	8
3	2

x	$f^{-1}(x)$
8	1
8	2
2	3

← no such function f^{-1}!

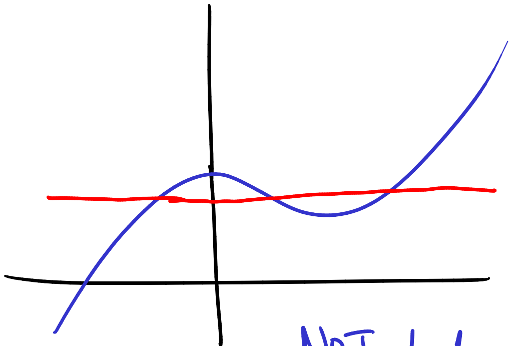
We say f is a **1-1 function** if

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.$$

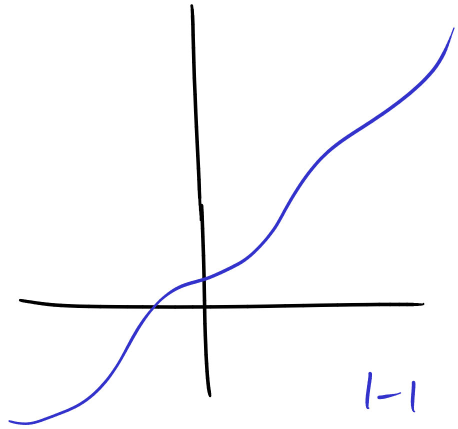
Fact if f is 1-1, then f^{-1} exists.

Horizontal line test

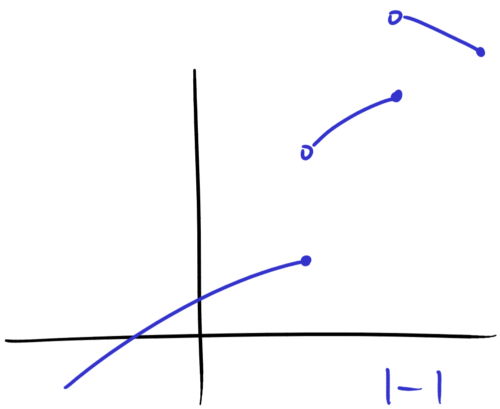
f is 1-1 just if no horizontal line meets the graph of f more than once.



NOT 1-1

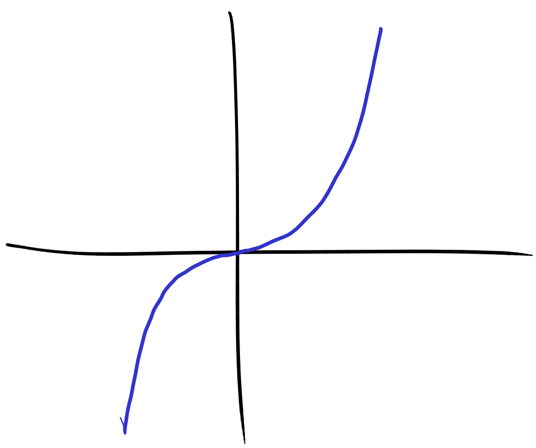


1-1



1-1

Ex $f(x) = x^3$



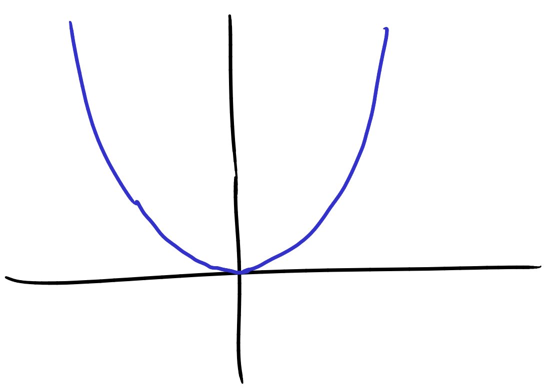
is 1-1.

Its inverse is

$$f^{-1}(x) = \sqrt[3]{x}.$$

(Because $\sqrt[3]{x^3} = x$ for all real x.)

Ex $f(x) = x^2$



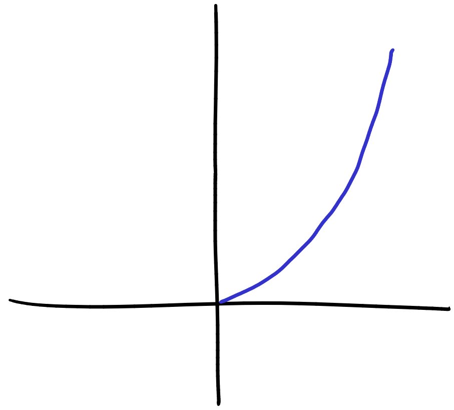
Not 1-1 on its "natural domain" $(-\infty, \infty)$.

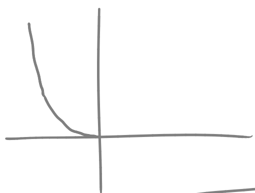
But if we take domain $[0, \infty)$

then f is 1-1 on this domain;

$\Rightarrow$ it has an inverse.

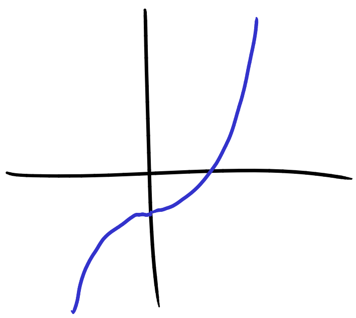
$$f^{-1}(x) = \sqrt{x}$$





$f^{-1}(x) = -\sqrt{x}$

<u>Ex</u> $f(x) = x^3 - 1$



To find $f^{-1}(x)$: <u>solve for x</u> in this equation

ie let $y = f(x)$

then $y = x^3 - 1$

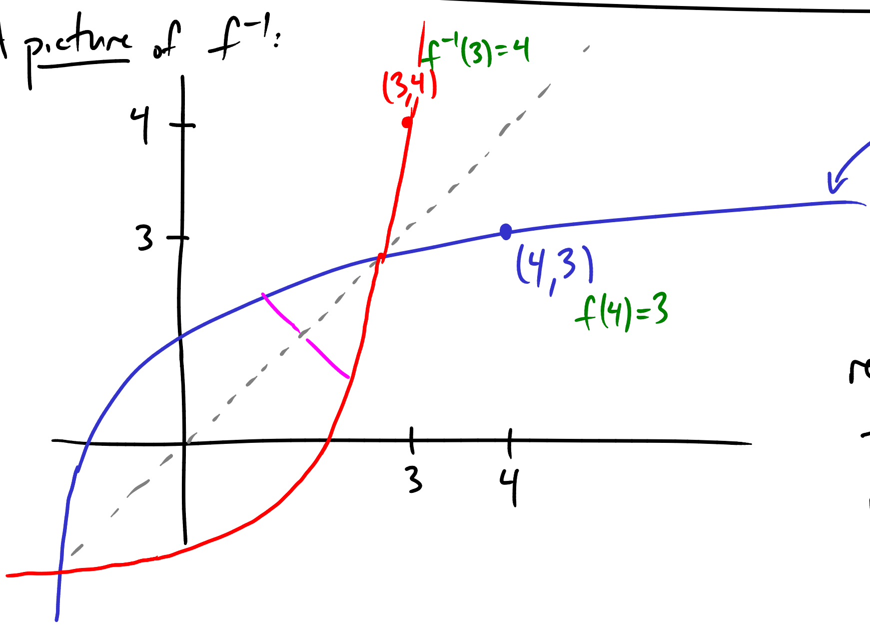
$y + 1 = x^3$

$x = \sqrt[3]{y+1}$

thus $f^{-1}(y) = \sqrt[3]{y+1}$

(or, $f^{-1}(x) = \sqrt[3]{x+1}$ or, $f^{-1}(t) = \sqrt[3]{t+1}$)

A <u>picture</u> of f^{-1}:

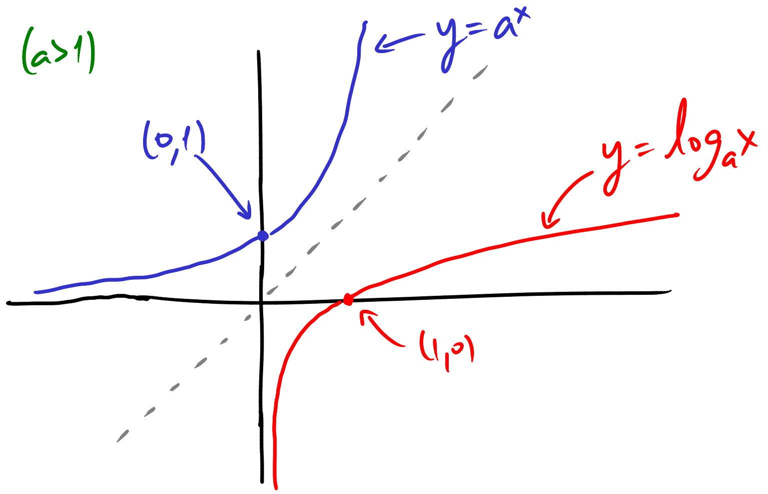


graph of f
$y = f(x)$

related by <u>reflection</u> in the line $y = x$,

ie <u>swapping</u> $y \longleftrightarrow x$.

$\underline{\text{Ex}}$ $f(x) = a^x$ is 1-1 if $a \neq 1$ $\quad a > 0$



We call its inverse $f^{-1}(x) = \log_a x$

domain: $(0, \infty)$

x	10^x
0	1
1	10
2	100
3	1000
4	10000
−1	0.1
−2	0.01

x	$\log_{10} x$
1	0
10	1
100	2
1000	3
10000	4
0.1	−1
0.01	−2

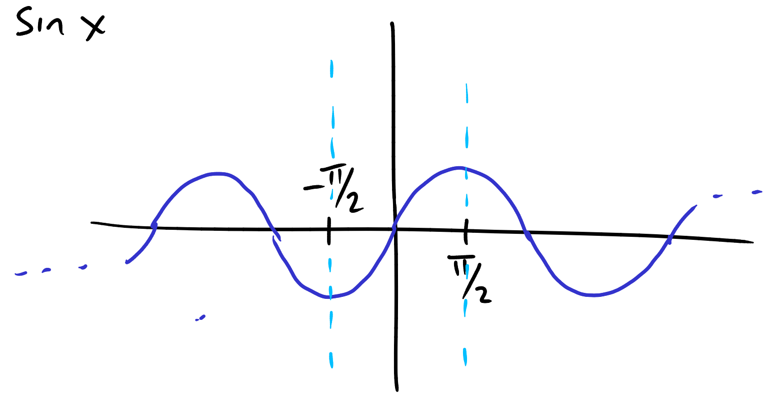
Laws of logarithms

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^r) = r \log_a x$

[Why? e.g. ① comes from $a^x a^y = a^{x+y}$]

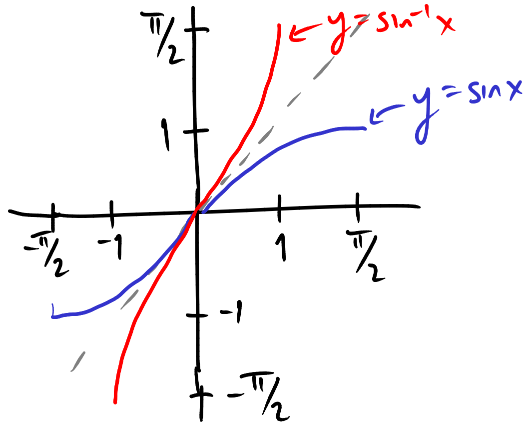
$\underline{\text{Ex}}$ $\log_3 63 - \log_3 7 = \log_3 \frac{63}{7} = \log_3 9 = 2$

$\underline{\text{Ex}}$ $f(x) = \sin x$



f is 1-1 on domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\qquad$ (not 1-1 on domain $(-\infty, \infty)$)

Let $\sin^{-1}$ mean the inverse of sin on domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



So in practice, "$\sin^{-1} x$" means "an angle θ with $\sin\theta = x$ and $-\pi/2 \le \theta \le \pi/2$."

<u>Ex</u> $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ because $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ <u>and</u> $-\pi/2 \le \frac{\pi}{6} \le \frac{\pi}{2}$.