Last time: derivatives

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

or

\[ f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

or

\[ f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \]

We say \( f(x) \) is differentiable at \( a \) if \( f'(a) \) exists

(ie if \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \) exists)

Example:

\( f(x) = x^2 \) is differentiable at all real numbers \( a \), because

\[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \] exists (and equals \( 2a \))

Example: Where is \( f(x) = |x| \) differentiable?

If \( x > 0 \), \( f'(x) = 1 \) (exists)

If \( x < 0 \), \( f'(x) = -1 \) (exists)

If \( x = 0 \), let's look closer:

\[ f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h} \]

DNE
So $f$ is not differentiable at $x = 0$. (and is differentiable at all other $x$.)
(In general, graph of $f$ has sharp corner at $x \Rightarrow f$ is not differentiable at $x$.)

**Ex** Where is $f(x) = \begin{cases} 1 & x \neq 0 \\
2 & x = 0 \end{cases}$ differentiable?

At any $x \neq 0$, $f'(x) = 0$ (exists)
At $x = 0$,

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

Plug in small $h$: $\frac{f(h) - f(0)}{h} = \frac{1 - 2}{h} = \frac{-1}{h}$ DNE

$\lim_{h \to 0} -\frac{1}{h}$ DNE

So $f(x)$ is not differentiable at $x = 0$. (\(\Rightarrow\) $f(x)$ is differentiable at all $x$ except 0.)

In general, where $f$ is not continuous, $f$ is not differentiable.

**Ex**

\(f\) is differentiable except at \(-3, -2, -1, 3, 5\)

(Rec: At points where \(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \pm \infty\) we say $f$ is not differentiable)
Interpretation of $f'(x)$

1. If $x(t)$ is the position of an object at time $t$, then $x'(t)$ is the velocity of the object.

\[ v(t) \]

Example: An electron in a uniform electric field moves with

\[ x(t) = \frac{1}{2} t^2 \]

What is its velocity at time $t$?

\[ v(t) = x'(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} \]

\[ = \lim_{h \to 0} \frac{1}{2} (t+h)^2 - \frac{1}{2} t^2 \]

\[ = t \]

2. In general, if $t$ = time

$f'(t)$ is the rate of change of $f(t)$.

Example: If $V(t)$ = volume of water in Lake Travis at time $t$ (in gal)

$V'(t) = \text{rate of change of the volume}$ (in gal/sec)

We can also repeat:

\[ f''(x) = \text{derivative of } f'(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \]

"second derivative"

If $x(t)$ is position

\[ x'(t) = v(t) \text{ is velocity} \]

\[ x''(t) = v'(t) = a(t) \text{ is acceleration} \] (rate of change of the velocity)
Another notation:

\[
\frac{df}{dx} \quad \text{or} \quad \frac{d}{dx} f(x) \quad \text{means} \quad f'(x) \quad (\approx \frac{\Delta f}{\Delta x})
\]

\[
\frac{d^2f}{dx^2} \quad \text{or} \quad \frac{d^2}{dx^2} f(x) \quad \text{means} \quad f''(x)
\]

\[\vdots\]

\[
\frac{d^nf}{dx^n} \quad \text{or} \quad \frac{d^n}{dx^n} f(x) \quad \text{means} \quad f^{(n)}(x) = f^{\text{n times}}(x)
\]

\[
\begin{align*}
\mathbb{E}_x \quad & \quad \frac{d}{dx} (x^2) = 2x \\
& \quad \frac{d^2}{dx^2} (x^2) = \frac{d}{dx} (2x) = 2
\end{align*}
\]