Lecture 11

Midterm 1 next Mon (1 week from today)

Covers material from Lectures 1-11 (then trig derivatives)

To prepare: 1. go over + redo HW's (exam has ~12 Quest-like problems)
2. get extra probs from text sections we covered

HW DS will be due Tue night, not Mon night — contains material from Lecture 11 (exam style)

My office hr: today (2-3, RLM 9134)

Last time: deriv. of polynomials, powers, exponentials, product rule

\[ \frac{d}{dx} x^3 e^x = 3x^2 e^x + x^3 e^x \]

\( f \quad f' \quad f' \quad g \quad g' \)

\[ \frac{d}{dx} \left( \frac{f}{g} \right) = \frac{g f' - f g'}{g^2} \]

Quotient Rule: if \( f \) and \( g \) are differentiable at \( x \) and \( g(x) \neq 0 \)

\[ \frac{d}{dx} \left( \frac{1}{x} \right) = \frac{x \cdot 0 - 1 \cdot 1}{x^2} = -\frac{1}{x^2} \]

(check: \( \frac{d}{dx} \left( \frac{1}{x} \right) = \frac{d}{dx} (x^{-1}) = -1 \cdot x^{-2} = -\frac{1}{x^2} \))

\[ \frac{d}{dx} \left( \frac{x^2 + 3x + 5}{1 + x} \right) = \frac{d}{dx} \left( \frac{x^2 + 3x + 2}{1 + x} \right) = ? \]

\[ = \frac{(1+x)(2x+3) - (x^2 + 3x + 5)(1)}{(1+x)^2} \]

\[ = \frac{(2x^2 + 5x + 3) - (x^2 + 3x + 5)}{(1+x)^2} \]

\[ = \frac{x^2 + 2x + 1}{(1+x)^2} = \frac{(x+1)^2}{(x+1)^2} = 1 \]

(why?)
\[
\frac{d}{dx} \left( \frac{e^x}{x-1} \right) = \frac{(x-1)e^x - e^x(1)}{(x-1)^2} = \frac{xe^x - e^x - e^x}{(x-1)^2} = e^x \frac{x-2}{(x-1)^2}
\]

\[
\frac{d}{dx} \left( \frac{e^x + e^x x^3}{\sqrt{x}} \right) = \sqrt{x} \frac{d}{dx} \left( e^x + e^x x^3 \right) - (e^x + e^x x^3) \frac{d}{dx} \frac{1}{\sqrt{x}} \\
(x > 0)
\]

\[
= \sqrt{x} \left( e^x + e^x x^3 + e^x 3x^2 \right) - (e^x + e^x x^3) \frac{1}{2\sqrt{x}} \\
= \cdots
\]

OR, can just split it up:
\[
\frac{d}{dx} \left( \frac{e^x + e^x x^3}{\sqrt{x}} \right) = \frac{d}{dx} \left( \frac{e^x}{\sqrt{x}} + \frac{e^x x^3}{\sqrt{x}} \right) = \frac{d}{dx} \left( e^x \frac{1}{\sqrt{x}} + e^x x^\frac{3}{2} \right) = \cdots
\]

just use product rule!
\[
\left( \frac{d}{dx} e^x \right) \frac{5}{2} \neq (x e^x)^{\frac{5}{2}}
\]

Derivatives of Trig Functions

Need to work out a few limits 1st:

Fact \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \).

Why? \( \sin x = \text{opposite} \)

From the picture:
\[
\sin x < x < \tan x
\]

So \( \frac{\sin x}{x} < 1 \) and \( x < \tan x = \frac{\sin x}{\cos x} \)

\( \cos x < \frac{\sin x}{x} < 1 \)
So \[ \lim_{x \to 0} \frac{\sin x}{x} \leq \lim_{x \to 0} \frac{\sin x}{x} \leq \lim_{x \to 0} 1 \]
\[ 1 \leq \lim_{x \to 0} \frac{\sin x}{x} \leq 1 \]
So \[ \lim_{x \to 0} \frac{\sin x}{x} = 1. \]

**Fact** \[ \lim_{x \to 0} \frac{\cos x - 1}{x} = 0. \]

Why? \[ \lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} \]
\[ = \lim_{x \to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} \]
\[ = \lim_{x \to 0} \frac{-\sin^2 x}{x(\cos x + 1)} \]
\[ = \lim_{x \to 0} -\frac{\sin x}{x} \cdot \frac{\sin x}{\cos x + 1} \]
\[ = (-1) \cdot \left( \frac{0}{2} \right) = 0 \]

**Fact** \[ \frac{d}{dx} (\sin x) = \cos x. \]

Why? \[ \frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \]
\[ = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \]
\[ = \lim_{h \to 0} \sin x \cdot \left( \frac{\cos h - 1}{h} \right) + \cos x \cdot \left( \frac{\sin h}{h} \right) \]
\[ = \sin x \cdot 0 + \cos x \cdot 1 \]
\[ = \cos x \]

Ex \[ \frac{d}{dx} (x^4 \sin x) = 4x^3 \sin x + x^4 \cos x. \]
\[ = x^3 (4 \sin x + x \cos x) \]
Fact \[ \frac{d}{dx} \cos x = -\sin x. \] (similar derivation)

All other deriv. of trig functions follow from these:

Ex: \[ \frac{d}{dx} (\tan x) = \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x. \]

Summary

\[ \frac{d}{dx} \sin x = \cos x \quad \frac{d}{dx} \tan x = \sec^2 x \quad \frac{d}{dx} \sec x = \sec x \cdot \tan x \]

\[ \frac{d}{dx} \cos x = -\sin x \quad \frac{d}{dx} \cot x = -\csc^2 x \quad \frac{d}{dx} \csc x = -\csc x \cot x \]