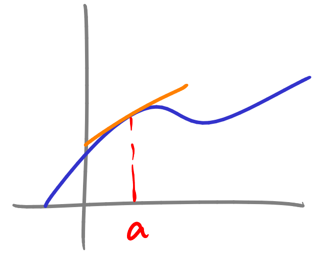
Lecture 18

12 Oct 2018

Last time: linearization for x near a, and f diff'ble at a,

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$



e.g. for x near 0, $\sin(x) \approx x$

for x near 0, $\ln(1+x) \approx x$

for x near 1, $\sqrt[3]{x} \approx 1 + \frac{x}{3}$ $\sqrt[n]{x} \approx 1 + \frac{x}{n}$

e.g. $\ln(1.07) \approx 0.07$ $\sqrt[10]{1.1} \approx 1.01$

Compound interest

Say you have \$100 in bank at 4%/yr interest.
Then how much will you have after n years?

"Compounded yearly":

1 year: $100 \times 1.04 = \$104$

2 years: $100 \times (1.04)^2 = \$108.16$

3 yrs: $100 \times (1.04)^3 = \$112.49$

$\vdots$

n yrs: $100 \times (1.04)^n$

"Compounded monthly":

1 month: $100 \times \left(1 + \frac{0.04}{12}\right)$

1 year: $100 \times \left(1 + \frac{0.04}{12}\right)^{12} = \$104.07$

n yrs: $100 \times \left(1 + \frac{0.04}{12}\right)^{12n}$

"Comp daily":

1 year: $100 \times \left(1 + \frac{0.04}{365}\right)^{365} = \$104.08$

n yrs: $100 \times \left(1 + \frac{0.04}{365}\right)^{365n}$

"Compounded continuously":

$\left(k = \#\text{ times to compound } \underline{\text{per year}}\right)$

1 year: $100 \times \lim_{k\to\infty}\left(1+\frac{0.04}{k}\right)^k$

n years. $100 \times \lim_{k\to\infty}\left(1+\frac{0.04}{k}\right)^{k\cdot n}$

$= 100 \times \left[\lim_{k\to\infty}\left(1+\frac{0.04}{k}\right)^k\right]^n$

using $\lim_{k\to\infty}\left(1+\frac{x}{k}\right)^k = e^x$

why is it true?

$= 100 \times \left(e^{0.04}\right)^n$

$= 100 \times e^{0.04n}$ exponential growth!

$\lim_{k\to\infty}\left(1+\frac{x}{k}\right)^k$

$= \lim_{k\to\infty} e^{\ln\left(1+\frac{x}{k}\right)\cdot k}$ use $\ln(1+t) \approx t$, t small

$= \lim_{k\to\infty} e^{\frac{x}{k}\cdot k} = \lim_{k\to\infty} e^x = e^x$

in general, could write

$$P(t) = P_0 \cdot e^{rt}$$

interest rate

amount of \$ at time t

starting amount

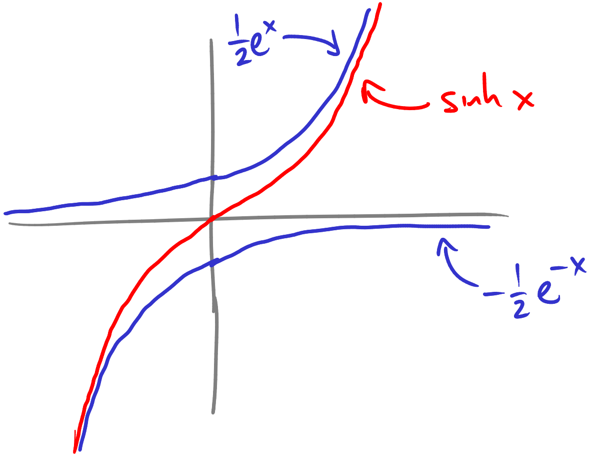
Hyperbolic functions

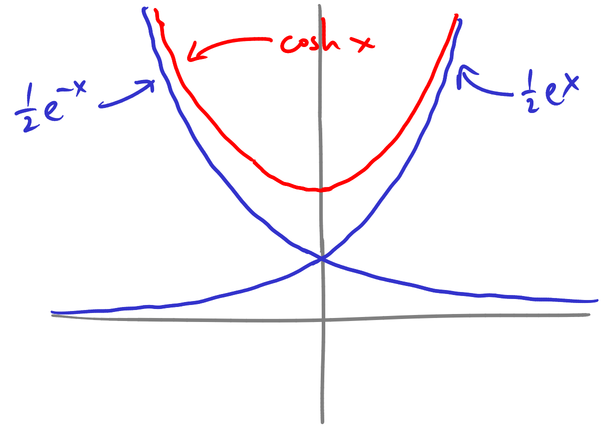
Hyperbolic functions are "cousins" of the usual trig functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$





Remark: $y = \cosh x$ is the shape of a freely-hanging heavy cord ("catenary")
(like telephone/power wire)

Hyperbolic identities

$$\sinh(-x) = -\sinh(x)$$
$$\cosh(-x) = \cosh(x)$$

$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Why? e.g. $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ $\quad \sinh(-x) = \dfrac{e^{-x} - e^{x}}{2} = -\sinh(x)$ ✓

e.g. $\cosh^2(x) - \sinh^2(x)$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

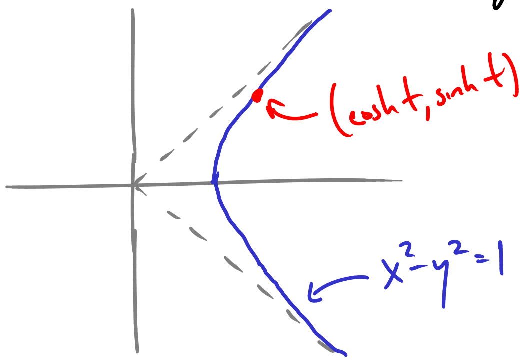
$$= \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4}$$

$$= \frac{e^{2x} + 2 \cdot e^x \cdot e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2 \cdot e^x \cdot e^{-x} + e^{-2x}}{4}$$

$$= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4} = \frac{4}{4} = 1.$$

So, $\cosh^2 t - \sinh^2 t = 1$: if $x = \cosh t$, $y = \sinh t$ then $x^2 - y^2 = 1$.

hyperbola



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