If \( f(x) \) has a local min at \( x = a \) and \( f(x) \) is differentiable at \( x = a \) and \( x = a \) is not an endpoint of the domain of \( f \), then \( f'(a) = 0 \) (hence tangent).

**Strategy for finding absolute max/min for a function \( f \) with domain \([a,b]\):**

1. Find values of \( f \) at all "critical numbers":
   - \( x \) where \( f'(x) = 0 \) or \( f'(x) \) DNE.
2. Find values of \( f(a), f(b) \)
3. Take max, min values of \( f \) from that list.

**Ex:** Find absolute max, min of \( f(x) = 12 + 4x - x^2 \) on \([0,5]\) \( (0 \leq x \leq 5) \)

1. \( f'(x) \) exists everywhere — no point where \( f'(x) \) DNE.
   - \( f'(x) = 4 - 2x \)
   - so \( f'(x) = 0 \) only at \( 4 - 2x = 0 \) \( \Rightarrow x = 2 \).
   - So only critical # is \( x = 2 \).
   - \( f(2) = 12 + 4 \cdot 2 - 2^2 = 16 \).
2. \( f(0) = 12 + 4 \cdot 0 - 0^2 = 12. \)
   - \( f(5) = 12 + 4 \cdot 5 - 5^2 = 7 \)
3. Max is \( f(2) = 16 \). Min is \( f(5) = 7 \).
Exercise
Find absolute max of
\[ f(x) = x^{-2} \ln x \quad \text{for} \quad 1 \leq x \leq 100. \]

1. \[ f'(x) = -2x^{-3} \ln x + x^{-2} \cdot \frac{1}{x} \]
   \[ = -2x^{-3} \ln x + x^{-3} \]
   \[ = x^{-3}(-2 \ln x + 1) \]
   \[ f'(x) = 0: \]
   \[ 0 = x^{-3}(-2 \ln x + 1) \quad \text{only if} \quad 0 = -2 \ln x + 1 \]
   \[ 2 \ln x = 1 \]
   \[ \ln x = \frac{1}{2} \]
   \[ x = e^{\frac{1}{2}} = \sqrt{e} \]
   \[ f(\sqrt{e}) = (\sqrt{e})^{-2} \ln (\sqrt{e}) = \frac{1}{e} \cdot \frac{1}{2} = \frac{1}{2e} \]

2. \[ f(1) = 1 - \ln(1) = 0 \]
   \[ f(100) = \frac{1}{10000} \ln 100 \]

3. The max is one of \[ \frac{1}{2e}, 0, \frac{\ln(100)}{10000} \].
   \[ \frac{1}{2e} > \frac{\ln(100)}{10000} \]
   So, \[ f'(x) = \frac{1}{2e} \]
   is abs max.
Graph using derivatives

How do we use \( f'(x) \) to get information about the graph of \( f(x) \)?

Ex. Find where the function \( f(x) = 3x^4 - 4x^3 - 12x^2 + 5 \) is increasing and where it is decreasing.

\[
f'(x) = 12x^3 - 12x^2 - 24x
\]

\[
= 12x(x^2 - x - 2)
\]

\[
= 12x(x-2)(x+1)
\]

To see if \( f'(x) \) is positive or negative, look at

<table>
<thead>
<tr>
<th>Sign of ( f'(x) )</th>
<th>+++++</th>
<th>+ + + +</th>
<th>+ + +</th>
<th>+ +</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>-</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

So \( f(x) \) is increasing for \( x \in (-1, 0) \cup (2, \infty) \)

decreasing for \( x \in (-\infty, -1) \cup (0, 2) \)

\[
f(-1) = 0
\]
\[
f(0) = 5
\]
\[
f(2) = -27
\]

Let's look closer at the critical pts. \( f'(x) = 0 \) at \( x = -1, 0, 2 \).
At $x = 2:$ $\frac{\text{sign } f'(x)}{2} \quad \begin{array}{c} \downarrow \\text{local min} \end{array}$

At $x = 0:$ $\begin{array}{c} \begin{array}{c} \downarrow \\text{local max} \end{array} \end{array}$

At $x = -1:$ $\begin{array}{c} \begin{array}{c} \downarrow \\text{local min} \end{array} \end{array}$

**First Derivative Test**

If $c$ is a critical number for $f(x),$

1. If $f'(x)$ changes sign from $+$ to $-$ at $c,$ then $f$ has local max at $c.$

2. If $f'(x)$ changes sign from $-$ to $+$ at $c,$ then $f$ has local min at $c.$

3. If $f'(x)$ doesn't change sign at $c,$ then $f$ has neither local max or local min at $c.$

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**Ex** Find all local max/min of $f(x) = \frac{1}{3}x(x+4)$ on $(0, \infty)$ and $(-\infty, 0)$

For $x \neq 0,$ $f'(x)$ exists everywhere.

$$f'(x) = \frac{1}{3}x^{\frac{2}{3}}(x+4) + x^{\frac{1}{3}}$$

$$= \frac{1}{3}x^{\frac{2}{3}} + \frac{4}{3}x^{\frac{2}{3}} + x^{\frac{1}{3}}$$

$$= \frac{4}{3}x^{\frac{2}{3}}(x+1)$$

$f'(x) = 0$ only at $x = -1.$

$s(x^{-\frac{1}{2}})^2 > 0$

on $(0, \infty):$ no local max/min
on \((-\infty, 0)\): \quad \text{sign of } f'(x) \quad +-- +
\begin{array}{c}
\times \\
-1 \\
0
\end{array}

S. \ x=-1 \ is \ \underline{\text{local minimum}}.