Our final exam set: Dec 17, 9-12 (in this room)
Next week I'm called for jury duty
Monday: Prof. Bob Gumpf

Mean Value Theorem

Suppose someone drives from Austin to San Antonio (80 mi)
in Austin at 1:30 pm
in San Antonio at 2:00 pm
Then they must have been speeding at some time between 1:30 and 2:00
because their average speed was \( \frac{80 \text{ mi}}{0.5 \text{ hr}} = 160 \text{ mph} \)

Fact: Suppose \( f \) is a function continuous on \([a, b]\)
differentiable on \((a, b)\)

Then there is some \( c \) in \((a, b)\)
such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \)

ie, there is some \( c \) in \((a, b)\) where the slope of the tangent line to graph \( y = f(x) \)
at \((c, f(c))\) equals the slope of the secant line connecting \((a, f(a))\) to \((b, f(b))\)!

Ex: \( f(x) = x^3 - x \)

MVT sez: there is some \( c \) in \((0, 2)\) such that \( f'(c) = 3 \)
Let's check: $f'(x) = 3x^2 - 1$
$f'(c) = 3 \Rightarrow 3c^2 - 1 = 3$
$3c^2 = 4$
$c^2 = \frac{4}{3}$
$c = \pm \frac{2}{\sqrt{3}}$
$c = \frac{2}{\sqrt{3}} \in (0, 2)$  \checkmark

It's important that $f$ is defined in $MV\overline{1}$!

---

Graphing using derivatives, cont'd

**Concavity**

Both of these have $f'(x) > 0$ for $x \in (a, b)$ but they are different:

- Graph of $y = f(x)$ is **concave up** if it lies **above** all its tangent lines in $(a, b)$
- **concave down** below $(a, b)$

---

**Fact**: If $f''(x) < 0$ for all $x \in (a, b)$ then graph of $y = f(x)$ is **concave down** on $(a, b)$

---

**Fact**: If $f''(x) > 0$ then graph of $y = f(x)$ is **concave up** on $(a, b)$
Ex \[ f(x) = x^2 \]
\[ f'(x) = 2x \]
\[ f''(x) = 2 \quad \text{so} \quad f''(x) > 0 \]
for all \( x \in (-\infty, \infty) \)

So, graph of \( y = x^2 \) is concave up
for all \( x \in (-\infty, \infty) \).

[Why? If \( f''(x) > 0 \) then slope of tangent line increases as \( x \) increases]

A point of inflection is a point \((c, f(c))\) where \( f \) is continuous and the graph \( y = f(x) \) changes from concave up to concave down or vice versa.

Ex \[ f(x) = x^3 - 3x \]
\[ f'(x) = 3x^2 - 3 = 3(x^2 - 1) \]
\[ f''(x) = 6x \]

So:
\[ x = 1 \quad \text{local min} \quad f(1) = -2 \]
\[ x = 0 \quad \text{inflexional} \quad f(0) = 0 \]
\[ x = -1 \quad \text{local max} \quad f(-1) = 2 \]

Second Derivative Test:
If \( f \) is twice differentiable at \( c \), \( f''(c) = 0 \), and
\[ 1. f''(c) > 0, \quad \text{then } c \text{ is local min} \]
\[ 2. f''(c) < 0, \quad \text{local max} \]
\[ 3. f''(c) = 0, \quad \text{then the test fails} \quad \text{no info} \]

Q: Find \( f(x) \) s.t.
for some \( c \), \( f'(c) = 0 \), \( f''(c) = 0 \)
but \( c \) is a local min!
\[ y = x^4 = f(x) \]

\[ f'(x) = 4x^3 \]
\[ f''(x) = 12x^2 \]

At \( x = 0 \),

\[ f(0) = 0 \quad f'(0) = 0 \quad f''(0) = 0 \quad f'''(0) = 0 \quad f^{(4)}(0) = 24 \]