Lecture 31

14 Nov 2018

Last time: calculating integrals using the FTC.

FTC: I. $\int_a^x f(t)\,dt$ (as a function of x) is an antiderivative of $f(x)$.

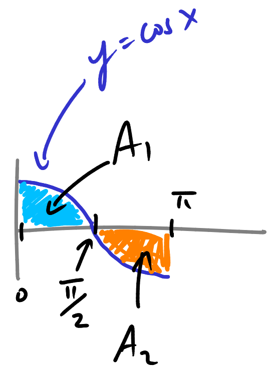
ie $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

II. $\int_a^b f(x)\,dx = F(b) - F(a) = F\Big|_a^b$ where $F(x)$ is any antideriv of $f(x)$.

[Exercise figure out how FTC II follows from FTC I.]

Ex $\frac{d}{dx}\int_{-412}^{2x} \cos(t^7)\,dt = 2\cos\left((2x)^7\right)$

Ex $\int_0^\pi \cos x\,dx = ?$ and what does it mean in terms of areas?



$\int_0^\pi \cos x\,dx = \sin x\Big|_0^\pi = \sin\pi - \sin 0 = 0 - 0 = 0$

ie $A_1 - A_2 = 0$ ie $A_1 = A_2$ ✓

Rk could also look at $\int_0^\pi |\cos x|\,dx$ to get the actual area $= A_1 + A_2$

— to do this, could find an antideriv. of $|\cos x|$

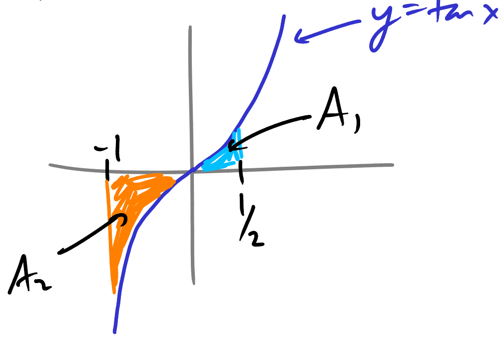
but could also just break the integral up into pieces

$$|\cos x| = \begin{cases} \cos x & 0 \le x \le \frac{\pi}{2} \\ -\cos x & \frac{\pi}{2} \le x \le \pi \end{cases}$$

so $\int_0^\pi |\cos x|\,dx = \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx$

$= \cdots = 1 + 1 = 2.$

Ex Is $\int_{-1}^{1/2} \tan x \, dx$ positive, negative or zero?



Q Can we get antideriv. of tan as $\dfrac{(\text{antideriv. of } \sin)}{(\text{antideriv. of } \cos)}$?

A: no

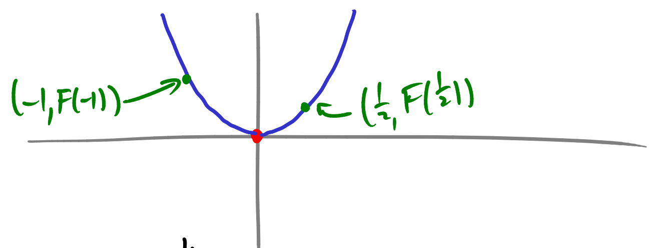
$$\int_{-1}^{1/2} \tan x \, dx = A_1 - A_2$$

but $A_2 > A_1$ so this is **negative**

Ex If $F(x) = \int_0^x \tan t \, dt$ sketch the graph of $F(x)$.

$F'' > 0$, $F' < 0$, F decr. | $F'' > 0$, $F' > 0$, F incr.

x, 0



$$F(0) = \int_0^0 \tan t \, dt = 0.$$

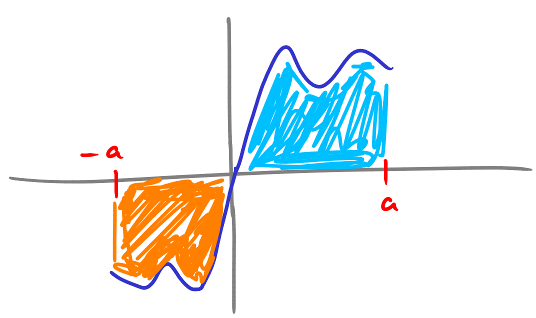
Rk
$$\int_{-1}^{1/2} \tan t \, dt = \int_0^{1/2} + \int_{-1}^0$$
$$= \int_0^{1/2} - \int_0^{-1} = F(\tfrac{1}{2}) - F(-1)$$

Ex Is $\int_{-1/2}^{1/2} \tan x \, dx$ positive, negative or zero? It's zero.

General rule: integrals of symmetric functions

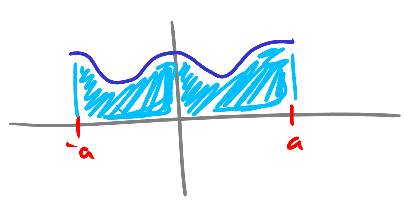
a) if f is odd, $f(-x) = -f(x)$

then $\int_{-a}^{a} f(x) \, dx = 0$



b) if f is even, $f(-x) = f(x)$

then $\int_{-a}^{a} f(x) \, dx = 2 \int_0^a f(x) \, dx$



Ex $\displaystyle\int_{-0.154}^{0.154} \frac{\overset{\text{odd}}{(\tan x)}\,\overset{\text{even}}{(x^6+29x^4+77x^2+961.2)}}{\underbrace{x^{12}+7\cos(32x)+5}_{\text{even}}}\,dx = 0$

$F = \dfrac{f \cdot g}{h}$

$F(-x) = \dfrac{f(-x)\cdot g(-x)}{h(-x)} = \dfrac{(-f(x))\cdot g(x)}{h(x)}$

$= -F(x)$

Indefinite integrals

Notation: $\int f(x)\,dx$ means: any antiderivative of $f(x)$.

Ex $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$ $\qquad \displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

$\int \sin x\,dx = -\cos x + C$ $\qquad \int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$ $\qquad \int \csc^2 x\,dx = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$ $\qquad \int \csc x \cot x\,dx = -\csc x + C$

Ex Find $\int 10x^4 + 6\sec^2 x\,dx$.

$= 10\cdot\frac{x^5}{5} + 6\tan x + C = \underline{2x^5 + 6\tan x + C}$

Ex Find $\displaystyle\int_0^{\pi/4} 10x^4 + 6\sec^2 x\,dx$

$= 2x^5 + 6\tan x \Big|_0^{\pi/4}$

$= \left(2\left(\frac{\pi}{4}\right)^5 + 6\tan\left(\frac{\pi}{4}\right)\right) - \left(2(0)^5 + 6\tan(0)\right)$

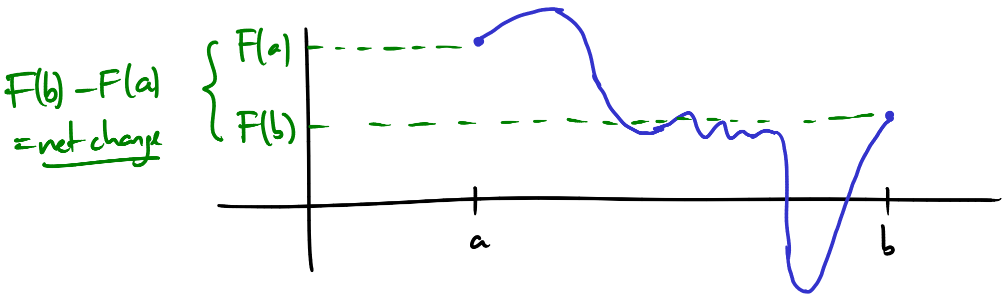
$= \left(\frac{\pi^5}{512} + 6\right) - 0 \qquad = \underline{\frac{\pi^5}{512} + 6}$

Net change

Given a function $F(t)$ $t = \text{time}$

$F'(t)$ is the rate of change of $F(t)$.

$$\int_a^b F'(t)\,dt = F(b) - F(a) = \text{net change of } F \text{ over time interval } [a,b].$$



Ex Water flows into a reservoir at the rate $(10t+6)$ ft^3/sec. (t in sec)

The reservoir contains 400 ft^3 of water at time $t=0$.

How much does it contain at time $t = 10$ s?

The net change from $t=0$ to $t=10$ is

$$\begin{aligned}
\int_0^{10} (10t+6)\,dt &= 5t^2 + 6t \Big|_0^{10} \\
&= \left(5(10)^2 + 6(10)\right) - \left(5\cdot 0^2 + 6\cdot 0\right) \\
&= 560 - 0 \\
&= 560\ ft^3
\end{aligned}$$

So the amount at time $t = 10$ s is $400 + 560\ ft^3 = \underline{960\ ft^3}$.