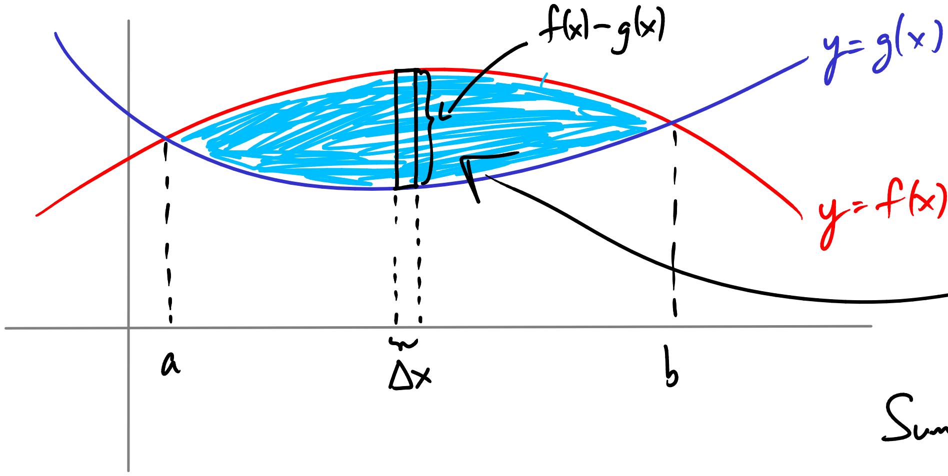
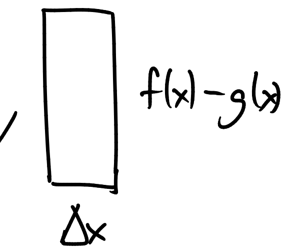
Lecture 35

28 Nov 2018

Last time: area between curves

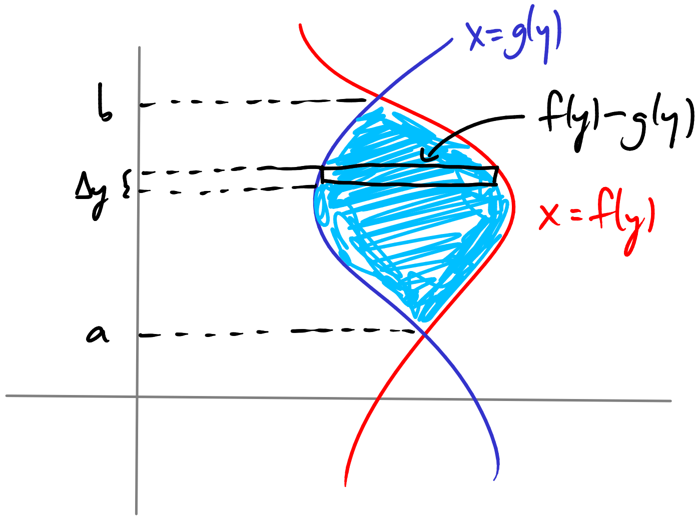


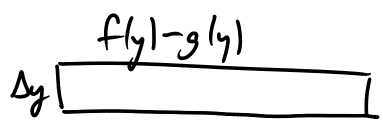


rect. area $= \Delta x\,(f(x) - g(x))$

Sum up all the rectangles:

total area $= \int_a^b dx\,(f(x) - g(x))$





rect. area $= \Delta y\,(f(y) - g(y))$

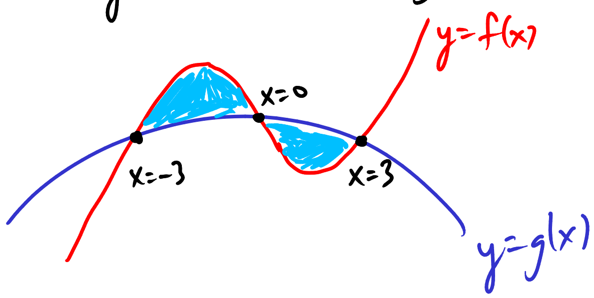
Sum up all the rectangles:

total area $= \int_a^b dy\,(f(y) - g(y))$

Ex Find the area of the region between

$$y = x^3 - x^2 - 7x - 4 = f(x)$$

$$y = -x^2 + 2x - 4 = g(x)$$



Points of intersection:

$$x^3 - x^2 - 7x - 4 = -x^2 + 2x - 4$$

$$x^3 - 9x = 0$$

$$x(x^2 - 9) = 0$$

$$x(x+3)(x-3) = 0 \longrightarrow x = 0, 3, -3$$

$$A = \int_{-3}^{0} f(x) - g(x)\,dx + \int_0^3 g(x) - f(x)\,dx$$

$$= \int_{-3}^{0} (x^3 - x^2 - 7x - 4) - (-x^2 + 2x - 4)\,dx + \quad \cdots$$

$$= \int_{-3}^{0} x^3 - 9x \, dx + \int_{0}^{3} -x^3 + 9x \, dx$$

$$= \left.\frac{x^4}{4} - \frac{9x^2}{2}\right|_{-3}^{0} + \left.-\frac{x^4}{4} + \frac{9x^2}{2}\right|_{0}^{3}$$

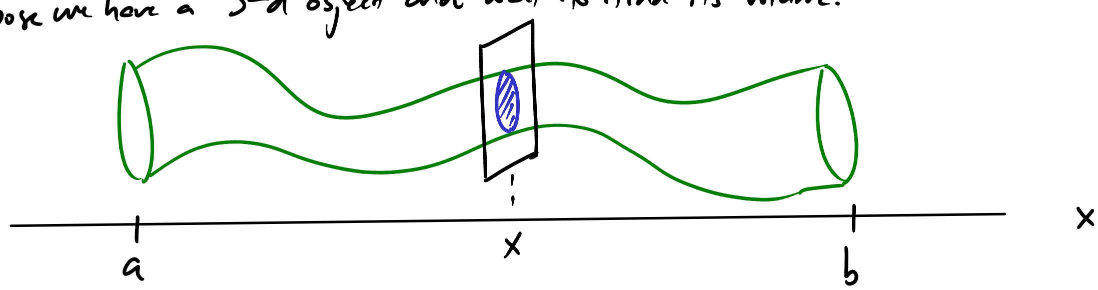
$$= 0 - \left(\frac{(-3)^4}{4} - \frac{9(-3)^2}{2}\right) + \left(-\frac{3^4}{4} + \frac{9\cdot 3^2}{2}\right) - 0$$

$$= -\frac{81}{4} + \frac{81}{2} + -\frac{81}{4} + \frac{81}{2}$$

$$= \frac{81}{2}$$

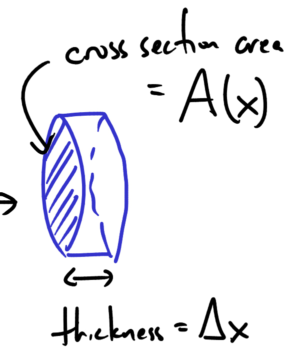
Volumes

Suppose we have a 3-d object and want to find its volume.



Chop the object into thin slices by knives at fixed closely-spaced values of x.

Each slice looks like →

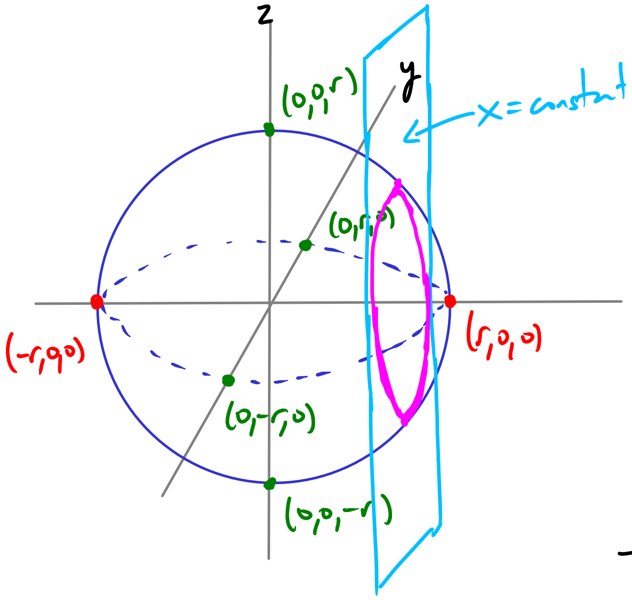


Volume of each slice: $V = A(x) \cdot \Delta x$

To get the total volume of our object, add up the slices:

$$V = \int_a^b A(x)\, dx$$

Ex Calculate the volume of a sphere of radius r.



Slice the sphere by planes $x = \text{constant}$.

Cross section = circle

What is its area? $A = \pi \cdot (\text{radius})^2$

radius of the cross-sectional circle, not the whole sphere!

this will depend on x

The sphere is defined by the equation $x^2 + y^2 + z^2 = r^2$

Holding x fixed, get cross section $y^2 + z^2 = r^2 - x^2$

this is a circle of radius $= \sqrt{r^2 - x^2}$

its area is $A = \pi \cdot \left(\sqrt{r^2 - x^2}\right)^2$

$$= \pi \cdot (r^2 - x^2)$$

Volume of sphere:

$$V = \int_{-r}^{r} A(x)\, dx$$

$$= \int_{-r}^{r} \pi \cdot (r^2 - x^2)\, dx$$

$$= \pi \cdot \left(r^2 x - \frac{x^3}{3} \Big|_{-r}^{r} \right)$$

$$= \pi \cdot \left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right) \right)$$

$$= \pi \cdot \left(\frac{2r^3}{3} + \frac{2r^3}{3} \right)$$

$$= \underline{\underline{\frac{4\pi r^3}{3}}}$$