Midterm 3  Fri 7 Dec
covers Lectures 24-36

Last time: computing volume by integration

**Average values**

What do we mean by the average value of some function $f$?

E.g. "average temperature over a day" — $f(t) =$ temperature (in °F) at time $t$ (in s)

What do we do to $f$ to get the average?

Average of a finite collection of numbers:

- Average of $\{2,4\} = \frac{2+4}{2} = \frac{6}{2} = 3$
- $\{2,4,7\} = \frac{2+4+7}{3} = \frac{13}{3}$
- $\{y_1, y_2, \ldots, y_n\} = \frac{y_1 + y_2 + \ldots + y_n}{n} = \frac{1}{n} \sum_{i=1}^{n} y_i$

To define average of a function $f(x)$ on the domain $[a,b]$: divide interval into $n$ parts

\[ a \quad x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_{n-1} \quad b \]

Take the average of the sample values: $y_i = f(x_i^*)$

So the approximate average of $f$ is

\[ \frac{y_1 + y_2 + \ldots + y_n}{n} = \frac{f(x_1^*) + f(x_2^*) + \ldots + f(x_n^*)}{n} \]

\[ = \sum_{i=1}^{n} f(x_i^*) \cdot \left( \frac{1}{n} \right) \]

Looks like a Riemann sum!

If we wanted to evaluate $\int_{a}^{b} f(x) \, dx$ we would write Riemann sums

\[ \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x \]

\[ = \sum_{i=1}^{n} f(x_i^*) \cdot \left( \frac{b-a}{n} \right) \]

Comparing the two:

\[ \int_{a}^{b} f(x) \, dx = (b-a) \cdot \text{(the average value of $f(x)$ on interval $[a,b]$)} \]
The average value of \( f(x) \) on the interval \([a,b]\) is \( \frac{1}{b-a} \int_a^b f(x) \, dx \).

**Example**

The average value of \( f(x) = \sin x \) on \([0, \pi]\) is

\[
\frac{1}{\pi-0} \int_0^\pi \sin x \, dx
\]

\[
= \frac{1}{\pi} \left(-\cos x \bigg|_0^\pi\right)
\]

\[
= \frac{1}{\pi} \left(-(-1) - (-1)\right) = \frac{1}{\pi} \cdot 2 = \frac{2}{\pi}
\]

**Example**

The average value of \( f(x) = c \) (constant) over \([a,b]\) is

\[
\frac{1}{b-a} \int_a^b c \, dx = \frac{1}{b-a} \cdot c \left(x \bigg|_a^b\right)
\]

\[
= \frac{1}{b-a} \cdot c(b-a) = \frac{c(b-a)}{b-a} = c
\]

**Example**

The average value of \( f(x) = x \) over \([0,10]\) is...

\[
\frac{1}{10-0} \int_0^{10} x \, dx = \frac{1}{10} \left(\frac{x^2}{2} \bigg|_0^{10}\right) = \frac{1}{10} \left(\frac{10^2}{2}\right) = \frac{1}{10} \cdot 50 = 5
\]

**Example**

The average value of \( f(x) = x^2 \) over \([0,10]\) is...

\[
\frac{1}{10-0} \int_0^{10} x^2 \, dx
\]

\[
= \frac{1}{10} \left(\frac{x^3}{3} \bigg|_0^{10}\right) = \frac{10^3}{3} \approx 33.3
\]
Example: What is the average value of \( f(x) = \sin^2 x \) over \([0, 2\pi]\)?

**First method:**

\[
\frac{1}{2\pi} \int_0^{2\pi} \sin^2 x \, dx
\]

Use trig identity:

\[
\cos 2x = -2 \sin^2 x + 1
\]
\[
\sin^2 x = \frac{1}{2} (1 - \cos 2x)
\]

So have:

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} (1 - \cos 2x) \, dx = \frac{1}{4\pi} \int_0^{2\pi} 1 - \cos 2x \, dx
\]

\[
= \frac{1}{4\pi} \left( x - \frac{1}{2} \sin 2x \right) \bigg|_0^{2\pi}
\]

\[
= \frac{1}{4\pi} \left( (2\pi - 0) - (0 - 0) \right) = \frac{1}{2}
\]

**Second method:**

\[
\sin^2 x + \cos^2 x = 1
\]

So their averages should also add up to 1.

And their averages should be the same.

So both must be \( \frac{1}{2} \).