5.5 Substitution Rule

old:
$y = f(x)$
$dy = f'(x)\,dx$

Ex $\int \cos 7x\,dx$ $\quad t = 7x$
$dt = 7\,dx$

$\int \cos t \cdot \frac{1}{7}\,dt$

$\frac{1}{7}\int \cos t\,dt$

$\frac{1}{7}\sin t$

$\frac{1}{7}\sin 7x + C$

In general

$\int f[g(x)](g'(x)\,dx)$ $\quad$ Let $u = g(x)$
$du = g'(x)\,dx$

$\int f(u)\,du$

Ex $\displaystyle\int_0^{\sqrt{\pi}} x \cos(x^2)\,dx$

$\boxed{u = x^2}$

$du = 2x\,dx$

$\displaystyle\frac{1}{2}\int_0^{\pi} \cos u \, du$

x	u
0	0
$\sqrt{\pi}$	π

$\displaystyle\frac{1}{2} \sin u \Big|_0^{\pi} = 0$

Ex $\displaystyle\int \frac{x+2}{\sqrt{x^2+4x}}\,dx$

$u = x^2 + 4x$

$du = (2x+4)\,dx$

$du = 2(x+2)\,dx$

$\frac{1}{2}\,du = (x+2)\,dx$

$\displaystyle\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$\displaystyle\frac{1}{2}\int u^{-\frac{1}{2}}\,du =$

$\displaystyle\int \frac{\frac{1}{2}\,du}{\sqrt{u}}$

$u^{\frac{1}{2}}$

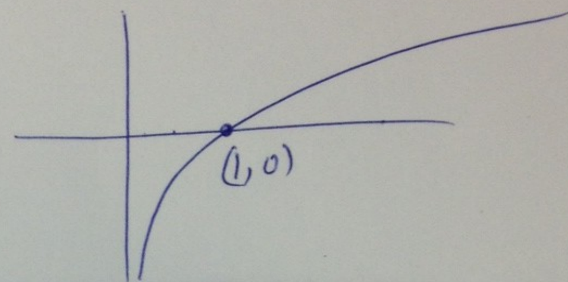
$\sqrt{x^2+4x} + C$

$$\int \frac{dx}{x \ln x} = \int \frac{1}{x \ln x}\, dx$$

$u = \ln x$

$du = \frac{1}{x}\, dx$

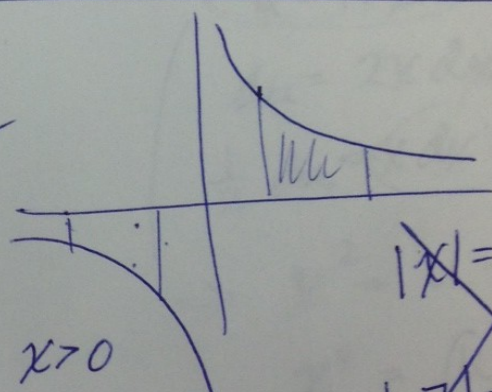
$$\int \frac{1}{u}\, du$$



$\ln|u|$

$\ln|\ln x| + C$

$$\int \frac{1}{x}\, dx = \ln|x| + C$$



$\cancel{|x| = x}$

$\cancel{|-x| = -x}$

$$f(x) = \ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{-1}{-x} = \frac{1}{x}, & x < 0 \end{cases}$$

$$\int e^{5x}\,dx \qquad u = 5x$$
$$du = 5\,dx$$

$$\frac{1}{5}\int e^u\,du$$

$$\frac{1}{5}e^u$$

$$\frac{1}{5}e^{5x} + C$$

$$\int \sqrt{1+x^2}\,\underbrace{x^5}_{x^4\cdot x}\,dx$$

$$u = 1+x^2$$
$$du = 2x\,dx$$
$$\frac{1}{2}\,du = x\,dx$$
$$x^2 = u-1$$
$$x^4 = (u-1)^2$$

$$\cancel{t = u-1}$$
$$\cancel{dt = du}$$

$$\frac{1}{2}\int \sqrt{u}\,(u-1)^2\,du$$

$$\frac{1}{2}\int u^{\frac{1}{2}}(u-1)^2\,du$$

$$\frac{1}{2}\int u^{\frac{1}{2}}(u^2-2u+1)\,du$$

$$\frac{1}{2}\int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$$

etc.

$$\int \tan x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$\int \frac{\sin x}{\cos x} \, dx$$

$$-\int \frac{1}{u} \, du$$

$$-\ln|u|$$

$$-\ln|\cos x|$$

$$\ln|\cos x|^{-1}$$

$$\boxed{\ln|\sec x| + C = \int \tan x \, dx}$$

$$\int \sec x \, dx \cdot \frac{\sec x + \tan x}{\sec x + \tan x}$$

$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\, dx$

$$\int \frac{1}{u} \, du$$

$\ln|u|$

$$\boxed{\ln|\sec x + \tan x| + C = \int \sec x \, dx}$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$$

$u = \sqrt{x} = x^{\frac{1}{2}}$

$du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2du = \frac{1}{\sqrt{x}}\,dx$

$$2\int e^{u}\,du$$

$$2e^{u}$$

$$2e^{\sqrt{x}} + C$$

$$\int \frac{dx}{\sqrt{x}\,(1+x)}$$

$$\cancel{u = 1+x}$$
$$\cancel{du = dx}$$
$$\cancel{x = u-1}$$
$$\cancel{\sqrt{x} = \sqrt{u-1}}$$
$$\cancel{\int \frac{1}{u\sqrt{u-1}}\,du}$$

$$\int \frac{2\cancel{u}\,du}{\cancel{u}\,(1+u^2)}$$

$$u = \sqrt{x}$$
$$u^2 = x$$
$$2u\,du = dx$$

$$2\int \frac{1}{1+u^2}\,du$$

$$2\tan^{-1} u$$

$$2\tan^{-1}\sqrt{x} + C$$

In general, if $\sqrt{ax+b}$

Try $u = \sqrt{ax+b}$

$$u^2 = ax+b$$

$$2u\,du = a\,dx$$

$$\int \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx$$

$u = \frac{\pi}{x}$

$du = -\frac{\pi}{x^2}\,dx$

$-\frac{du}{\pi} = \frac{1}{x^2}\,dx$

$$-\frac{1}{\pi}\int \cos u \, du$$

$$-\frac{1}{\pi}(+\sin u)$$

$$-\frac{1}{\pi}\sin\left(\frac{\pi}{x}\right) + C$$

$$\int \tan^2\theta \sec^2\theta \, d\theta$$

$u = \tan\theta$

$du = \sec^2\theta \, d\theta$

$$\int u^2 \, du$$

$$\frac{u^3}{3}$$

$$\frac{\tan^3\theta}{3} + C$$

$$\int \frac{5}{x^2+6x+10}\,dx$$

$$\cancel{u = x^2+6x+10}$$
$$\cancel{du = (2x+6)\,dx}$$

[illegible]

$$\cancel{u = x^2}$$
$$\cancel{du = 2x\,dx}$$

Use completing the square on $x^2+6x+10$

$$= \underbrace{x^2+6x+9}\,\underbrace{-9+10}$$

$$= (x+3)^2+1$$

$$5\int \frac{1}{(x+3)^2+1}\,dx$$

$$u = x+3$$
$$du = dx$$

$$5\int \frac{1}{u^2+1}\,du$$

$$5\tan^{-1} u$$

$$5\tan^{-1}(x+3) + C$$

$$I = \int \frac{\sqrt{x}}{x+1}\,dx$$

$\cancel{u = x+1}$
$\cancel{x = u-1}$
$\cancel{du = dx}$
$\cancel{\sqrt{x} = \sqrt{u-1}}$

$$\cancel{\int \frac{\sqrt{u-1}}{u}\,du}$$

$u = \sqrt{x}$
$u^2 = x$
$2u\,du = dx$

$$I = \int \frac{u}{u^2+1}\,2u\,du$$

$$2\int \frac{u^2}{u^2+1}\,du$$

If $\frac{\text{poly}}{\text{poly}}$ w/ deg num ≥ deg denom

long division.

$$\frac{u^2}{u^2+1} = \frac{u^2+1-1}{u^2+1}$$

$$= 1 - \frac{1}{u^2+1}$$

$$\begin{array}{r} 1 \\ u^2+1 \overline{)\, u^2 } \\ \underline{u^2+1} \\ -1 \end{array}$$

$$\frac{u^2}{u^2+1} = 1 - \frac{1}{u^2+1}$$

$$I = 2\int\left(1 - \frac{1}{u^2+1}\right)du$$

$$= 2u - 2\tan^{-1}u$$

$$= 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C.$$