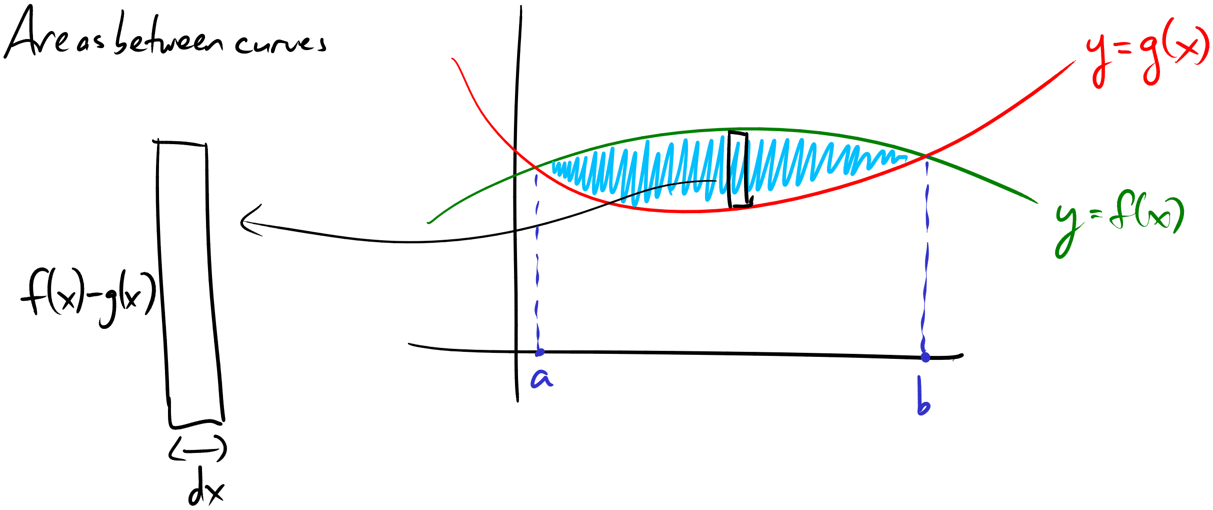
Lecture 7 — 18 Sep 2012

Last time: Areas between curves

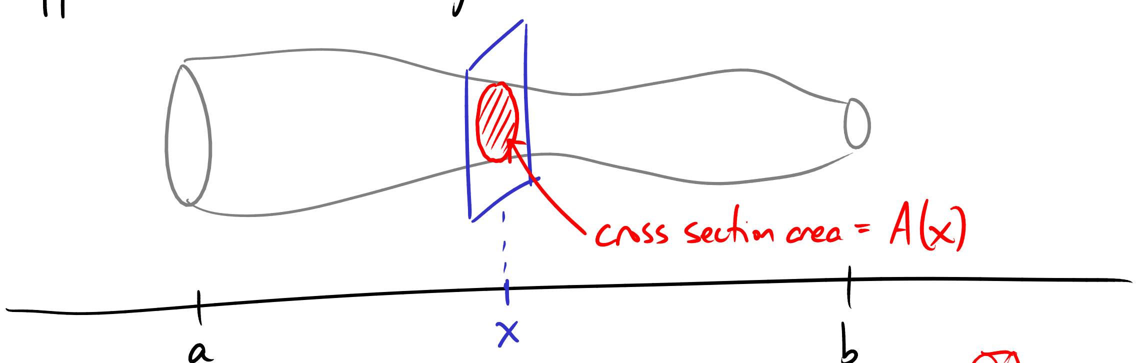


rectangle has $A = (f(x)-g(x))\,dx$

sum them up: total area $= \int_a^b (f(x)-g(x))\,dx$

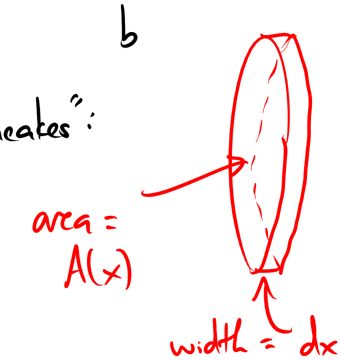
Volumes (Ch 6.2)

Suppose we have some 3-d object and want to find its volume.



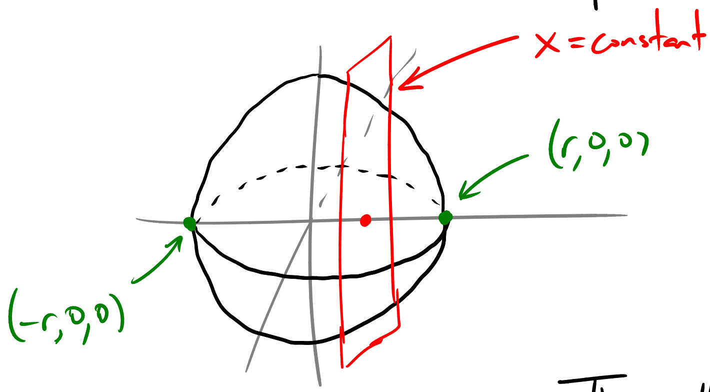
Chop the object into slices which look like "pancakes":

volume of each slice is $A(x)\,dx$



To get the whole volume, add up the slices: $V = \int_a^b A(x)\,dx$

Ex Calculate the volume of a sphere of radius r.



Slice the sphere by plane $x = \text{constant}$.

Sphere is $x^2+y^2+z^2 \leq r^2$

At fixed value of x:

this is $y^2+z^2 \leq r^2-x^2$

This is the inside of a circle with radius $= \sqrt{r^2-x^2}$.

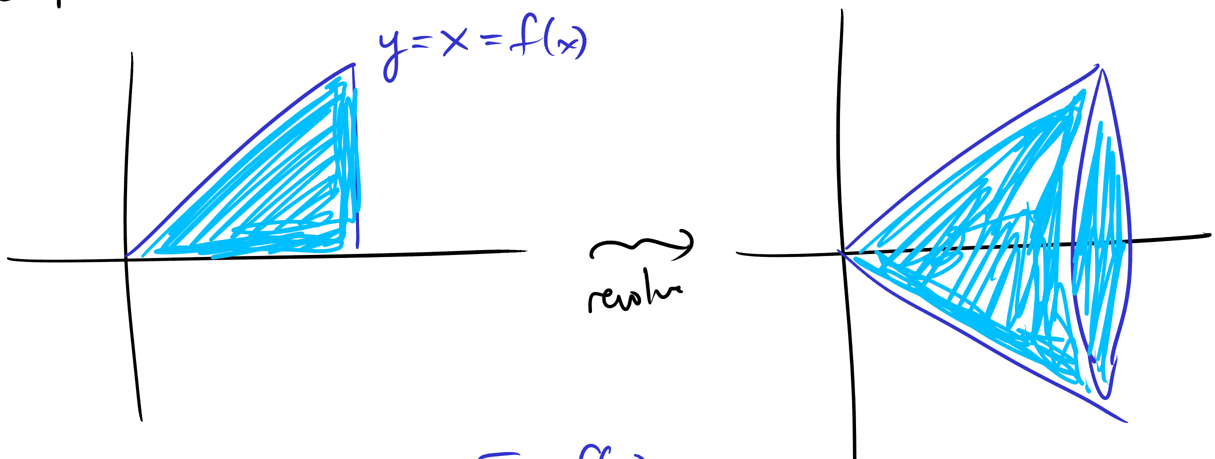
So the cross sections are circles, with area $A(x) = \pi\left(\sqrt{r^2-x^2}\right)^2$.

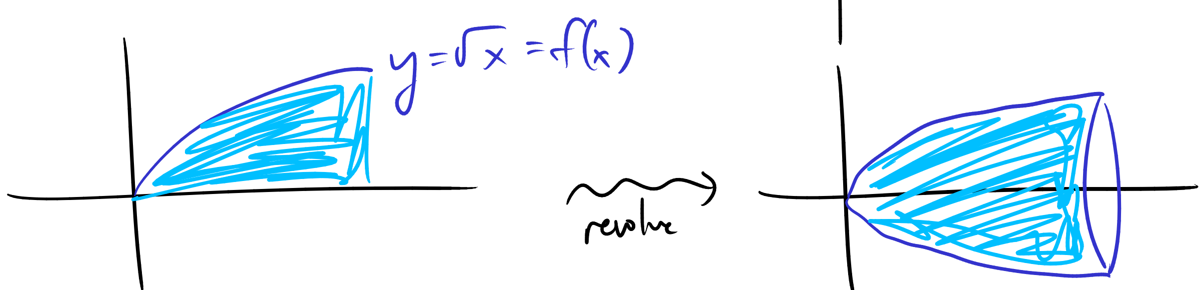
$$= \pi(r^2-x^2).$$

Volume of sphere:
$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2-x^2)\,dx$$
$$= \pi\left(r^2x - \tfrac{1}{3}x^3\right)\Big]_{-r}^{r}$$
$$= \underline{\underline{\tfrac{4}{3}\pi r^3}}$$

A common type of solid: "solid of revolution" — take the region under some graph and revolve it around, say, the x-axis.

Ex





The cross-section area of such a solid is just $A(x) = \pi f(x)^2$.
(Because the cross-section is a circle with radius $f(x)$.)

Ex Find the volume of a solid obtained by revolving the area under $y = \sqrt{x}$ around the x-axis with x from 0 to 2.

$$V = \int_0^2 dx\, A(x) = \int_0^2 dx\, \pi\left(\sqrt{x}\right)^2 = \int_0^2 dx\, \pi x = \underline{\underline{2\pi}}$$

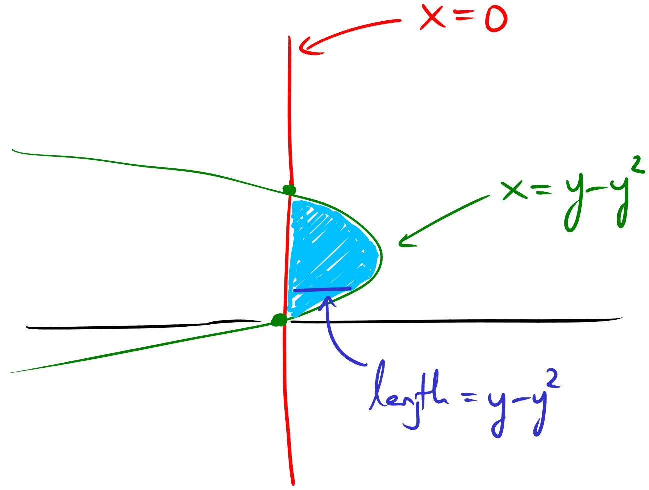
Can also revolve around, say, the y-axis.

Ex Find the volume of a region obtained by revolving the region between

$x = y - y^2$

$x = 0$

around the y-axis.



Intersections: $y - y^2 = 0$

$y(1-y) = 0$

$y = 0$ or $y = 1$

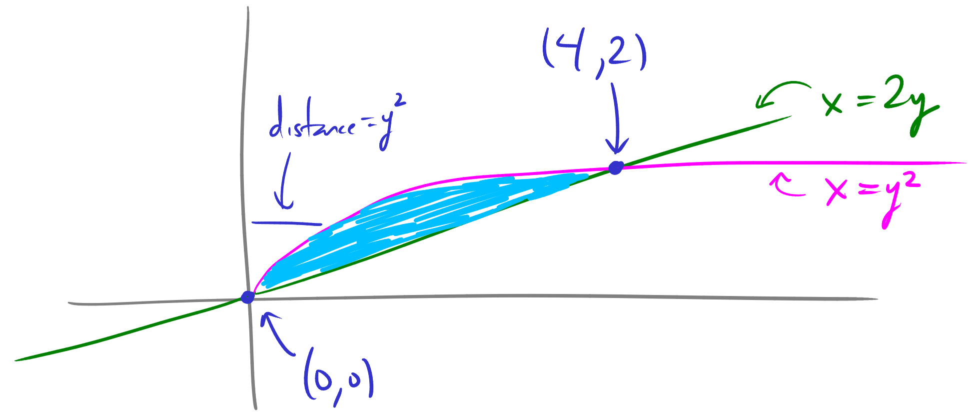
$\longrightarrow$ ints. at $(0,0)$ and $(0,1)$

[and at $y = \frac{1}{2}$, parabola has $x = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4} > 0$]

$$V = \int_0^1 dy\, A(y) = \int_0^1 dy\, \pi\left(y - y^2\right)^2 = \cdots = \underline{\underline{\frac{\pi}{30}}}$$

We may also encounter solids which can be sliced into little "washers"
(circular disc with a circular hole cut out.)

Ex Let R be the region between $y=\sqrt{x}$ and $x=2y$.
Find the volume of the solid obtained by rotating R around the y-axis.



Intersection:

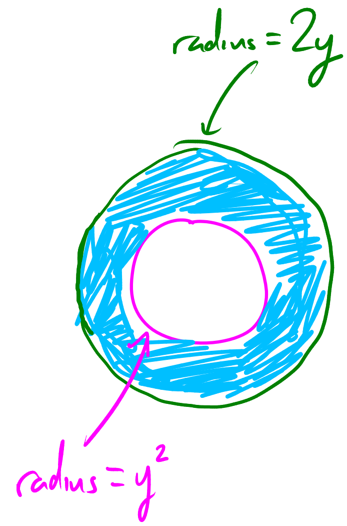
$y=\sqrt{x} \implies x=y^2$

also $\quad x=2y$

so $\quad 2y=y^2$

$y^2-2y=0$

$y(y-2)=0 \implies y=0$ or $y=2$



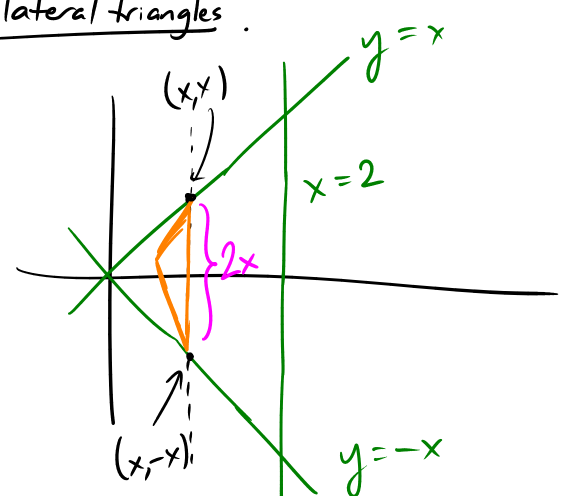
Rotate R around y axis: cross sections look like washers.
Radii determined by distance from y-axis, i.e. value of $\underline{\underline{x}}$.

$$A(y)=\pi(2y)^2-\pi(y^2)^2$$

$$=\pi(4y^2-y^4)$$

$$V=\int_0^2 A(y)\,dy=\int_0^2 \pi(4y^2-y^4)\,dy=\cdots=\underline{\underline{\frac{64}{15}\pi}}$$

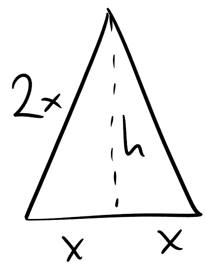
Ex Calculate the volume of a solid whose base is the region between $y=x$, $y=-x$ and $x=2$, and whose cross sections at fixed x are <u>equilateral triangles</u>.



$$V=\int_0^2 A(x)\,dx$$

$A(x) =$ area of an equilateral triangle with side length $2x$

Can just plug into the formula for area of an equilateral triangle, if you know it. Otherwise, do this:



$$x^2 + h^2 = (2x)^2$$
$$h^2 = 3x^2$$
$$h = \sqrt{3}\,x$$

$$A = \frac{1}{2} b \cdot h = \frac{1}{2}(2x)(\sqrt{3}\,x)$$
$$= \sqrt{3}\,x^2$$

So $A(x) = \sqrt{3}\,x^2$

$$V = \int_0^2 A(x)\,dx = \int_0^2 \sqrt{3}\,x^2\,dx = \frac{1}{3}\sqrt{3}\,x^3\Big]_0^2 = \underline{\underline{\frac{8}{3}\sqrt{3}}}$$