Lecture 9 21 Sep 2012

(More) trigonometric integrals (Ch 7.2)

$$\int \sin^5\theta \cos\theta \, d\theta$$

$u = \sin\theta$
$du = \cos\theta \, d\theta$

$$= \int u^5 \, du$$

$$= \frac{1}{6}u^6 + C = \frac{1}{6}\sin^6\theta + C$$

Similarly for $\int \sin^a\theta \cos\theta \, d\theta$ or $\int \cos^b\theta \sin\theta \, d\theta$

But: what about $\int \sin^3\theta \, d\theta$?

Use $\sin^2\theta + \cos^2\theta = 1$ so $\sin^2\theta = 1 - \cos^2\theta$

So $\int \sin^3\theta \, d\theta = \int \sin\theta \cdot \sin^2\theta \, d\theta$

$$= \int \sin\theta \cdot (1 - \cos^2\theta) \, d\theta$$

$$\left(= \int \sin\theta \, d\theta - \int \sin\theta \cos^2\theta \, d\theta \right)$$

$$= \int (1 - \cos^2\theta)(\sin\theta \, d\theta)$$

$u = \cos\theta$
$du = -\sin\theta \, d\theta$

$$= \int (1 - u^2)(-du)$$

$$= \int (u^2 - 1) \, du = \frac{1}{3}u^3 - u + C = \underline{\frac{1}{3}\cos^3\theta - \cos\theta + C}$$

Ex $\int \sin^5\theta \cos^2\theta \, d\theta$

$$= \int \underbrace{\sin^4\theta}_{?} \underbrace{\cos^2\theta}_{u^2} \underbrace{(\sin\theta \, d\theta)}_{-du}$$

$u = \cos\theta$
$du = -\sin\theta \, d\theta$
$-du = \sin\theta \, d\theta$

$$= \int (\sin^2\theta)^2 \cos^2\theta \, (\sin\theta \, d\theta)$$

$$= \int (1-\cos^2\theta)^2 \cos^2\theta \, (\sin\theta \, d\theta)$$

$$= \int (1-u^2)^2 u^2 \, (-du)$$

$$= \dots$$

$$= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C = \underline{-\frac{\cos^3\theta}{3} + 2\frac{\cos^5\theta}{5} - \frac{\cos^7\theta}{7} + C}$$

General rule for $\int \sin^a\theta \cos^b\theta \, d\theta$:

<u>If a is odd</u>, then pull out one factor $(\sin\theta)$, write $(\sin\theta \, d\theta)$, use $\sin^2\theta = 1-\cos^2\theta$ to eliminate the rest of the $(\sin\theta)$ factors, substitute $u = \cos\theta$.

<u>If b is odd</u>, then pull out one factor $(\cos\theta)$, write $(\cos\theta \, d\theta)$, use $\cos^2\theta = 1-\sin^2\theta$ to eliminate the rest of the $(\cos\theta)$ factors, substitute $u = \sin\theta$.

What about <u>even</u> powers?

Ex $\int \sin^2\theta \, d\theta$

$$= \int \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \int \frac{1}{2} - \frac{1}{2}\cos 2\theta \; d\theta$$

$$= \frac{1}{2}\left(\int 1 - \cos 2\theta \, d\theta \right)$$

$$= \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= \underline{\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C}$$

Half-angle formulas:

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$
$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

Ex $\int \cos^4\theta \, d\theta$

$$= \int (\cos^2\theta)^2 \, d\theta$$

$$= \int \left(\frac{1}{2}(1+\cos 2\theta)\right)^2 d\theta$$

$$= \frac{1}{4}\int 1 + 2\cos 2\theta + \cos^2 2\theta \;\; d\theta$$

$$= \frac{1}{4}\int 1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta) \; d\theta$$

$$= \frac{1}{4}\int \frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta \; d\theta$$

$$= \cdots$$

$$= \underline{\underline{\frac{3\theta}{8} + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + C}}$$

Ex $\int \tan^6 x \sec^4 x \, dx = ?$

Similar rules to what we used for sin, cos above: want $u = \tan x$

$du = \sec^2 x \, dx$

$$\int \underbrace{\tan^6 x}_{u^6} \underbrace{\sec^2 x}_{?} \underbrace{(\sec^2 x \, dx)}_{du}$$

Use $\sec^2 x = 1 + \tan^2 x$

$$\to \int = \int \tan^6 x \left(1+\tan^2 x\right) \left(\sec^2 x \, dx\right)$$

$$= \int u^6 (1+u^2) \, du$$

$$= \int u^6 + u^8 \, du$$

$$= \frac{1}{7}u^7 + \frac{1}{9}u^9 + C = \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$$

Same strategy works whenever sec x appears to an even power

Ex $\int_0^{\pi/4} \tan^3 x \sec^5 x \, dx$

$u = \sec x$

$du = \sec x \tan x$

$$= \int_0^{\pi/4} \tan^2 x \underbrace{\sec^4 x}_{u^4} \underbrace{(\tan x \sec x \, dx)}_{du}$$

use $\tan^2 x = \sec^2 x - 1$

$$\to \int = \int_0^{\pi/4} (\sec^2 x - 1) \sec^4 x \, (\tan x \sec x \, dx)$$

$$= \int_1^{\sqrt{2}} (u^2-1)\, u^4 \, du \qquad\qquad x = \pi/4 \longleftrightarrow u = \sqrt{2} = \frac{2}{\sqrt{2}}$$

$$x = 0 \longleftrightarrow u = 1$$

$$= \cdots = \underline{\underline{\frac{2}{35}\left(1+6\sqrt{2}\right)}}$$

Same strategy works for $\int \tan^a x \sec^b x$ whenever a is <u>odd</u> (and $b \geq 1$)

Handy facts:

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

<u>Ex</u> $\int \tan^3 x \, dx$

$$= \int \tan x \tan^2 x \, dx$$

$$= \int \tan x \left(\sec^2 x - 1\right) dx$$

$$= \underbrace{\int \tan x \sec^2 x}_{\left(\begin{array}{ll} u = \tan x & \Rightarrow \int u \, du \\ du = \sec^2 x & = \frac{1}{2}u^2 \end{array}\right)} - \int \tan x \, dx$$

$$= \frac{1}{2}\left(\tan^2 x\right) - \ln|\sec x| + C$$

Ex $\int \sin(4x)\cos(7x)\,dx$

Use product-sum identities:

$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$

$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

$$= \frac{1}{2}\int \sin(-3x) + \sin(11x)\,dx$$

$$= \frac{1}{2}\left(\frac{1}{3}\cos(3x) - \frac{1}{11}\cos(11x)\right) + C$$

$$= \underline{\underline{\frac{1}{6}\cos(3x) - \frac{1}{11}\cos(11x) + C}}$$