Lecture 13 | 1 Oct 2012

Last time: "strategy for integration"

Ex $\int x\sqrt{2-\sqrt{1-x^2}}\,dx$

$u = 1-x^2$ $\qquad du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$= -\frac{1}{2}\int\sqrt{2-\sqrt{u}}\,du$

$w = \sqrt{u}$

$w^2 = u$

$2w\,dw = du$

$= -\frac{1}{2}\int\sqrt{2-w}\;2w\,dw$

$z = 2-w \qquad w = 2-z$

$dz = -dw$

$= \int\sqrt{z}\,(2-z)\,dz$

$= \int 2z^{1/2} - z^{3/2}\,dz \quad = \ldots$

Ex $\int \frac{3x^2-2}{x^2-2x-8}\,dx$ $\qquad$ long div then par frac

Ex $\int \frac{3x^2-2}{x^3-2x-8}\,dx$

$u = x^3-2x-8$

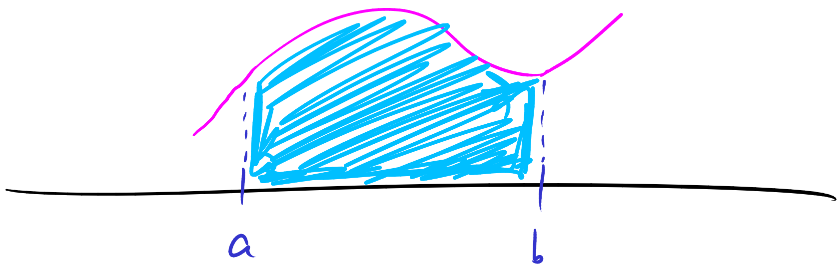
$du = 3x^2-2\,dx$

$= \int \frac{du}{u} \quad = \ldots$

Improper Integrals

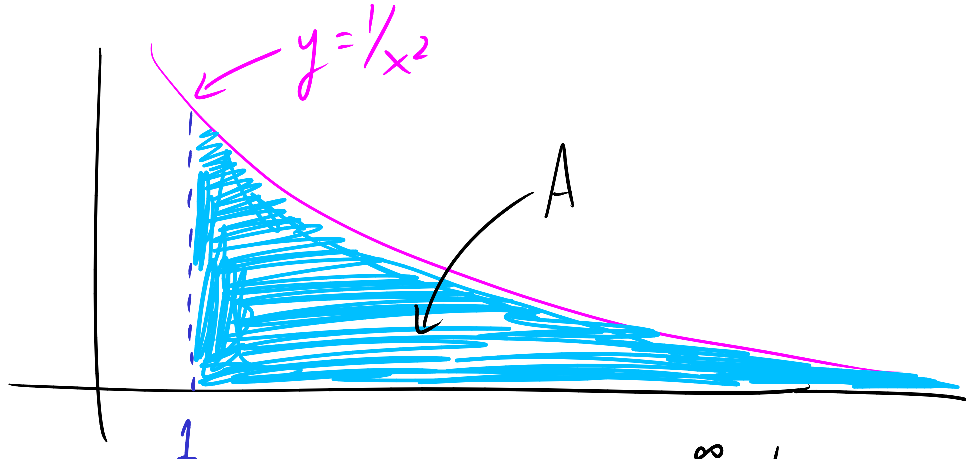
The definite $\int$ we've done so far were always of the form $\int_a^b f(x)\,dx$

where $f(x)$ was well defined, finite for every $a \le x \le b$.



These are called <u>proper</u> (definite) <u>integrals</u>.

How about:



the area of this <u>infinite</u> region? i.e $A = \int_1^\infty \frac{1}{x^2}\,dx$

This really means: define $A(t) = \int_1^t \frac{1}{x^2}\,dx$

then $A = \lim_{t\to\infty} A(t) = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx.$

So, let's calculate it:

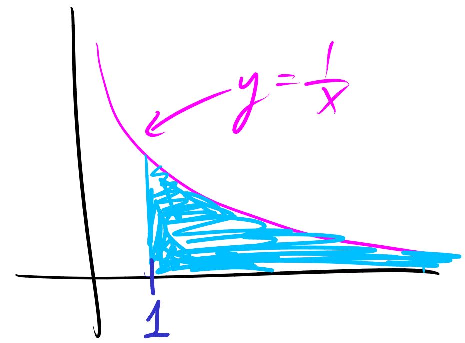
$$\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left(-\frac{1}{x}\Big]_1^t\right) = \lim_{t\to\infty} \left(1-\frac{1}{t}\right) = \underline{\underline{1}}$$

ie $\underline{\underline{A=1}}$

When the limit exists (as it did here), we say the improper integral is <u>convergent</u>.

ie: $\int_1^\infty \frac{1}{x^2}\,dx$ is <u>convergent</u>, (and its value is 1.)

How about $\int_1^\infty \frac{1}{x}\,dx$?



$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \left(\ln|x| \Big|_1^t \right) = \lim_{t\to\infty} (\ln t)$$

This limit doesn't exist (or, goes to $+\infty$)

So, this improper integral is <u>not</u> convergent — it is <u>divergent</u>.

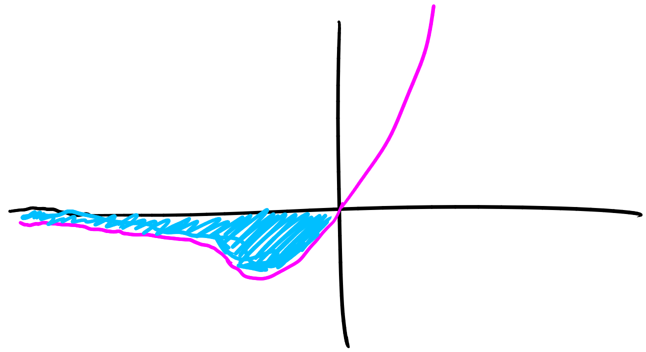
[If the limit doesn't exist, or goes to $+\infty$ or $-\infty$, we say the $\int$ is <u>divergent</u>]

So far we saw: $\int_1^\infty \frac{1}{x^2}\,dx$ is <u>convergent</u>

$\int_1^\infty \frac{1}{x}\,dx$ is <u>divergent</u>

General rule: for $a>0$, $\int_a^\infty \frac{1}{x^p}\,dx$ is $\begin{cases} \underline{\text{convergent}} \text{ if } p>1 \\ \underline{\text{divergent}} \text{ if } p\le 1 \end{cases}$

Ex $\displaystyle\int_{-\infty}^{0} x e^{x}\, dx$



$$= \lim_{t\to-\infty} \int_t^0 x e^x\, dx$$

Integrate by parts:

$$= \lim_{t\to-\infty} \left[-te^{t} - 1 + e^{t} \right]$$

$\infty \cdot 0$ $\qquad\qquad$ $\to 0$

$\Longrightarrow$ use L'Hospital:

$$\lim_{t\to-\infty} \frac{-t}{e^{-t}} = \lim_{t\to-\infty} \frac{-1}{-e^{-t}} = \lim_{t\to-\infty} e^{t} = 0$$

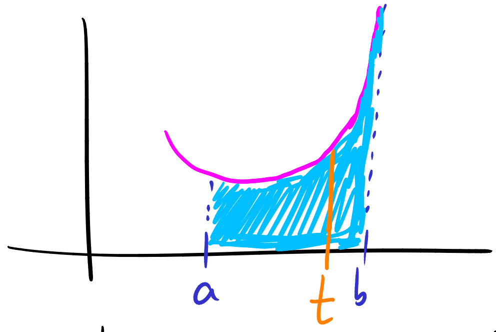
$$= \underline{\underline{-1}}$$

So altogether $\displaystyle\int_{-\infty}^{0} x e^{-x}\, dx = \underline{\underline{-1}}$ (convergent)

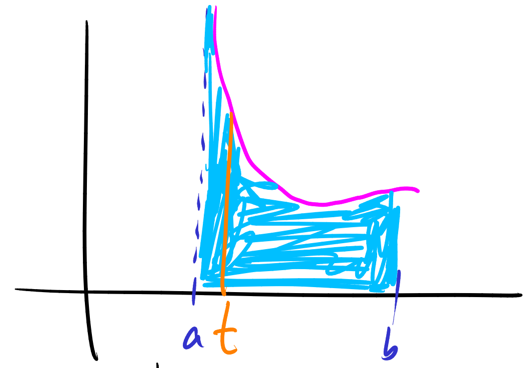
<u>Another</u> kind of improper $\int$:

$\displaystyle\int_a^b f(x)\, dx$ where $f(x)$ becomes <u>infinite</u> for some x

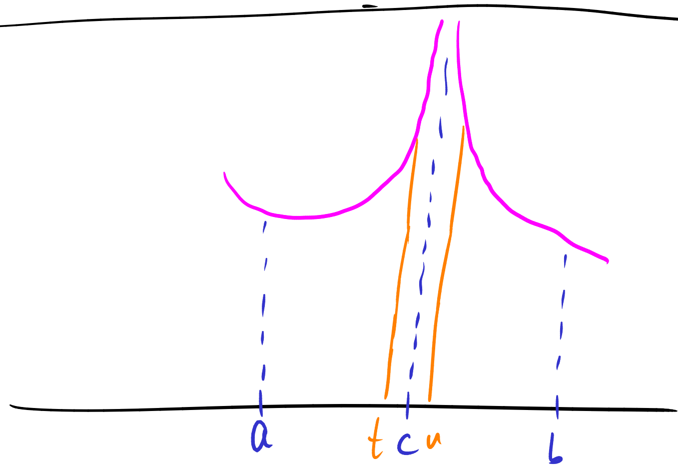
(i.e. $f(x)$ has a <u>vertical asymptote</u>)



Here $\int_a^b f(x)\,dx$ <u>means</u> $\lim_{t\to b^-}\int_a^t f(x)\,dx$



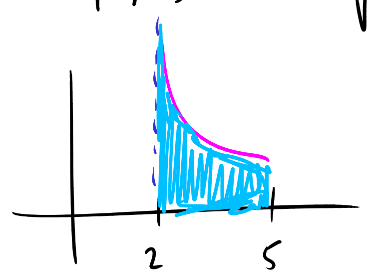
Here $\int_a^b f(x)\,dx$ <u>means</u> $\lim_{t\to a^+}\int_t^b f(x)\,dx$



Here $\int_a^b f(x)\,dx$ means

$$\left(\lim_{t\to c^-}\int_a^t f(x)\,dx\right)+\left(\lim_{u\to c^+}\int_u^b f(x)\,dx\right)$$

<u>Ex</u> $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ <u>improper</u>, because $\frac{1}{\sqrt{x-2}}$ goes to ∞ as $x\to 2^+$.



So $\displaystyle\int = \lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}\,dx$ (by u-sub: $u=x-2$)

$\displaystyle = \lim_{t\to 2^+}\left(2\left(\sqrt{3}-\sqrt{t-2}\right)\right)$

$= \underline{2\sqrt{3}}$ (<u>convergent</u>)

A general rule:

$$\int_0^a \frac{1}{x^p}\,dx \quad \text{is} \quad \begin{cases} \underline{\text{convergent}} \text{ if } p<1 \\ \underline{\text{divergent}} \text{ if } p\geq 1 \end{cases}$$