Lecture 32

14 Nov 2012

Last time: Power series (centered at a)

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

We think of x as a variable
c_n as constants
a as constant

Ex

$$\sum_{n=0}^{\infty} \frac{x^n}{n^2+1} = 1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots$$ centered at 0

$$\sum_{n=0}^{\infty} x^{2n+3} \cdot (3+n) = 3x^3 + 4x^5 + 5x^7 + 6x^9 + \cdots$$ centered at 0

$$\sum_{n=0}^{\infty} (2x+2)^n \cdot \frac{1}{n!} = 1 + (2x+2) + \frac{(2x+2)^2}{2} + \frac{(2x+2)^3}{6} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{n!} (x+1)^n = 1 + 2(x+1) + 2(x+1)^2 + \frac{4}{3}(x+1)^3 + \cdots$$ centered at -1

Main question about power series: For which values of x does the series converge?

We define the interval of convergence I to be the set of all x for which the series converges.

It can be:

1) $I = \{a\}$

2) $I = (-\infty, \infty)$

3) $I = (a-R, a+R)$ $\quad (a-R, a+R]$
$\quad [a-R, a+R)$ $\quad [a-R, a+R]$

In case 1) we say "$R=0$"
2) we say "$R=\infty$"

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ Power series centered at $a=-2$.

$$a_n = \frac{n(x+2)^n}{3^{n+1}}$$

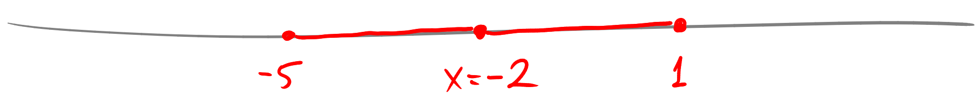
Ratio test: $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}}{n(x+2)^n}\right| \cdot \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}}\right|$$

$$= \left|\frac{3^{n+1}}{3^{n+2}}\right| \cdot \left|\frac{n+1}{n}\right| \cdot \left|\frac{(x+2)^{n+1}}{(x+2)^n}\right|$$

$$= \frac{1}{3} \cdot \underbrace{\left(\frac{n+1}{n}\right)}_{\to 1 \text{ as } n\to\infty} \cdot |x+2|$$

So $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}|x+2|$$

So, $\sum$ conv. if $\frac{1}{3}|x+2| < 1$ i.e. $|x+2| < 3$
div. if $\frac{1}{3}|x+2| > 1$ i.e. $|x+2| > 3$ so $\underline{R=3}$



Check convergence at endpoints:

$x=1$: $\displaystyle\sum_{n=1}^{\infty} \frac{n\cdot(1+2)^n}{3^{n+1}} = \sum_{n=1}^{\infty} n\cdot\frac{3^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{n}{3}$ diverges by TFD

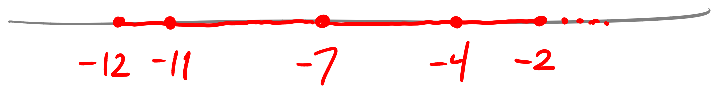
$x=-5$: $\displaystyle\sum_{n=1}^{\infty} \frac{n\cdot(-5+2)^n}{3^{n+1}} = \sum_{n=1}^{\infty} n\cdot\frac{(-3)^n}{3^{n+1}} = \sum_{n=1}^{\infty} (-1)^n\cdot\frac{n}{3}$ div. by TFD

So, interval of convergence is $\quad I = (-5, 1)$

Ex Suppose $\displaystyle\sum_{n=0}^{\infty} c_n (x+7)^n$ converges at $x = -2$.

- Does it also converge at $x = -4$?

  Yes: it's centered at -7 and int. of conv. contains $x = -2$, i.e.



  $R \geq 5$, so the int. of conv. also contains $x = -4$. $\quad (|-7-(-4)| = 3 < R)$

- Does it also converge at $x = -11$?

  Yes: $|-7-(-11)| = 4 < R$ since $R \geq 5$

- Does it also conv. at $x = -13$? $\quad$ Don't know.

Ex $\displaystyle\sum_{n=0}^{\infty} \frac{(5x-4)^n}{n^3}$

Ratio Test: $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n^3}{(5x-4)^n}\right| \cdot \left|\frac{(5x-4)^{n+1}}{(n+1)^3}\right|$

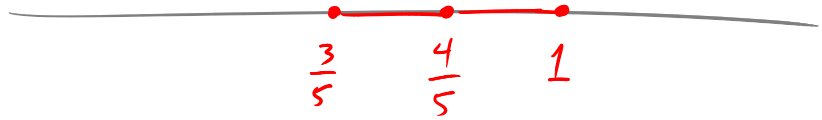
$$= \frac{n^3}{(n+1)^3} \cdot \left|\frac{(5x-4)^{n+1}}{(5x-4)^n}\right|$$

$$= \underbrace{\frac{n^3}{(n+1)^3}}_{\to 1} \cdot |5x-4|$$

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |5x-4| \implies$ $\sum$ converges if $|5x-4| < 1$ i.e. $|x - \frac{4}{5}| < \frac{1}{5}$

diverges if $|5x-4| > 1$ i.e. $|x - \frac{4}{5}| > \frac{1}{5}$

So $R = \frac{1}{5}$, $a = \frac{4}{5}$



Check endpoints:

$x = 1$: $\sum_{n=0}^{\infty} \frac{1}{n^3}$ conv. by p-test

$x = \frac{3}{5}$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^3}$ conv. by Alt Series Test

So $I = \left[\frac{3}{5}, 1\right]$