Lecture 36

28 Nov 2012

Housekeeping:

Notes at www.ma.utexas.edu/users/neitzke/teaching/408L
(Includes a PDF w/ all notes)

Message of the last few lectures: every (sufficiently nice) function is a series!

Uses of Taylor series

Taylor series for $f(x)$ centered at a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(x) \qquad \text{for } |x-a|<R$$

We know a bunch of examples (in particular, every power series we studied so far was an example):

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad |x|<1$$

$$\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n} \qquad |x|<1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{all } x$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{all } x$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{all } x$$

Taylor polynomial of f of degree d, centered at a:

$$T_d(x) = \sum_{n=0}^{d} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(first few terms of Taylor series: $\sum$ up to d instead of ∞)

Ex Use the Taylor polynomial of degree 2, centered at 0, for e^x to estimate $\sqrt[4]{e}$.

$$\sqrt[4]{e} = e^{1/4}$$

We could use the formula above for $T_2(x)$, but we already know the Taylor series for e^x, so just use that:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \underbrace{1 + x + \frac{x^2}{2}}_{T_2(x)} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$T_2(x) = 1 + x + \frac{x^2}{2}.$$

To get $\sqrt[4]{e}$, just plug in $x = \frac{1}{4}$:

$$T_2\left(\frac{1}{4}\right) = 1 + \frac{1}{4} + \frac{\left(\frac{1}{4}\right)^2}{2} = 1 + \frac{1}{4} + \frac{1}{32} = \frac{41}{32}$$

Ex Use the Taylor polynomial of degree 3 for $\sin(x)$ centered at 0 to estimate $\sin\left(\frac{1}{10}\right)$.

We know Taylor series for $\sin(x)$:

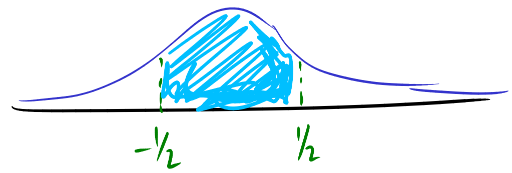
$$\sin(x) = \underbrace{x - \frac{x^3}{3!}}_{T_3(x)} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$T_3(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}$$

Plug in $x = \frac{1}{10}$: $T_3\left(\frac{1}{10}\right) = \frac{1}{10} - \frac{\left(\frac{1}{10}\right)^3}{6} = \frac{1}{10} - \frac{1}{6000} = \frac{599}{6000}$

Ex Use the Taylor polynomial for e^{-x^2} of degree 2 centered at 0 to estimate

$$\int_{-1/2}^{1/2} e^{-x^2}\,dx.$$



To get Taylor poly for e^{-x^2}:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

so $$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{6} + \cdots$$

$$= \underbrace{1 - x^2}_{T_2(x)} + \frac{x^4}{2} - \frac{x^6}{6} + \cdots$$

So estimate for $\int_{-1/2}^{1/2} e^{-x^2}\,dx$ is $\int_{-1/2}^{1/2} T_2(x)\,dx$

$$= \int_{-1/2}^{1/2} 1 - x^2\,dx$$

$$= x - \frac{1}{3}x^3 \Big]_{-1/2}^{1/2}$$

$$= \left(\frac{1}{2} - \frac{1}{3}\cdot\frac{1}{8}\right) - \left(-\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{8}\right)$$

$$= 1 - \frac{1}{12} = \frac{11}{12}$$

Another use of Taylor series:

Ex Calculate $\sum_{n=0}^{\infty} \left(-\frac{\pi^2}{16}\right)^n \cdot \frac{1}{(2n)!}$

Idea: this looks a bit like the Taylor series for $\cos(x)$...

$$\sum_{n=0}^{\infty} \left(-\frac{\pi^2}{16}\right)^n \frac{1}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\pi/4\right)^{2n}}{(2n)!}$$

But $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

So $\sum (-1)^n \frac{\left(\pi/4\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{4}\right) = \underline{\underline{\frac{\sqrt{2}}{2}}}$

Ex Find $\int_0^t \ln(1+x^3)\, dx$ as a power series.

Remember $\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}$ and $\ln(1+x^3) = \ln(1-(-x^3))$

$$\ln(1+x^3) = \sum_{n=1}^{\infty} -\frac{(-x^3)^n}{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n}}{n} \quad \text{for } |x|<1$$

So $\int_0^t \ln(1+x^3)\, dx = \int_0^t \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n}}{n}\, dx$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} \left.\frac{x^{3n+1}}{3n+1}\right]_{x=0}^{x=t}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \left(\frac{t^{3n+1}}{n(3n+1)} - \frac{0^{3n+1}}{n(3n+1)}\right)$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{3n+1}}{n(3n+1)}$$

Ex Find the Taylor poly. of degree 1 for $\sqrt[3]{x}$ centered at $a = 27$.

Taylor poly.

$$T_1(x) = f(a) + \frac{f'(a)}{1!}(x-a)$$

$f(x) = \sqrt[3]{x} = x^{1/3}$ $\qquad f(a) = f(27) = 3$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\qquad f'(a) = f'(27) = \frac{1}{3}(27)^{-2/3} = \frac{1}{27}$

So $T_1(x) = \underline{3 + \frac{1}{27}(x-27)}$