## Housekeeping:

HW02 due 3am 1/26 (tonight!)
HW01 due 3am 1/29 (this Fri)

**See me:**

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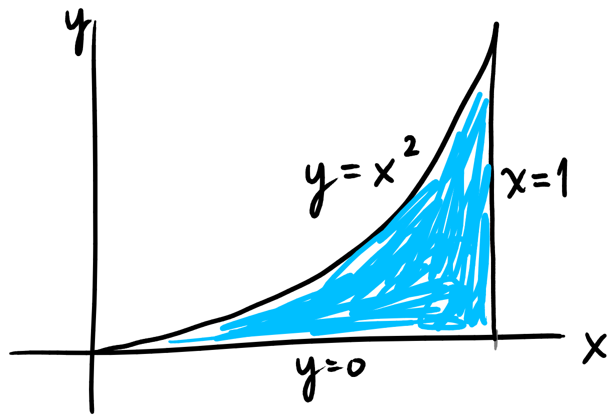
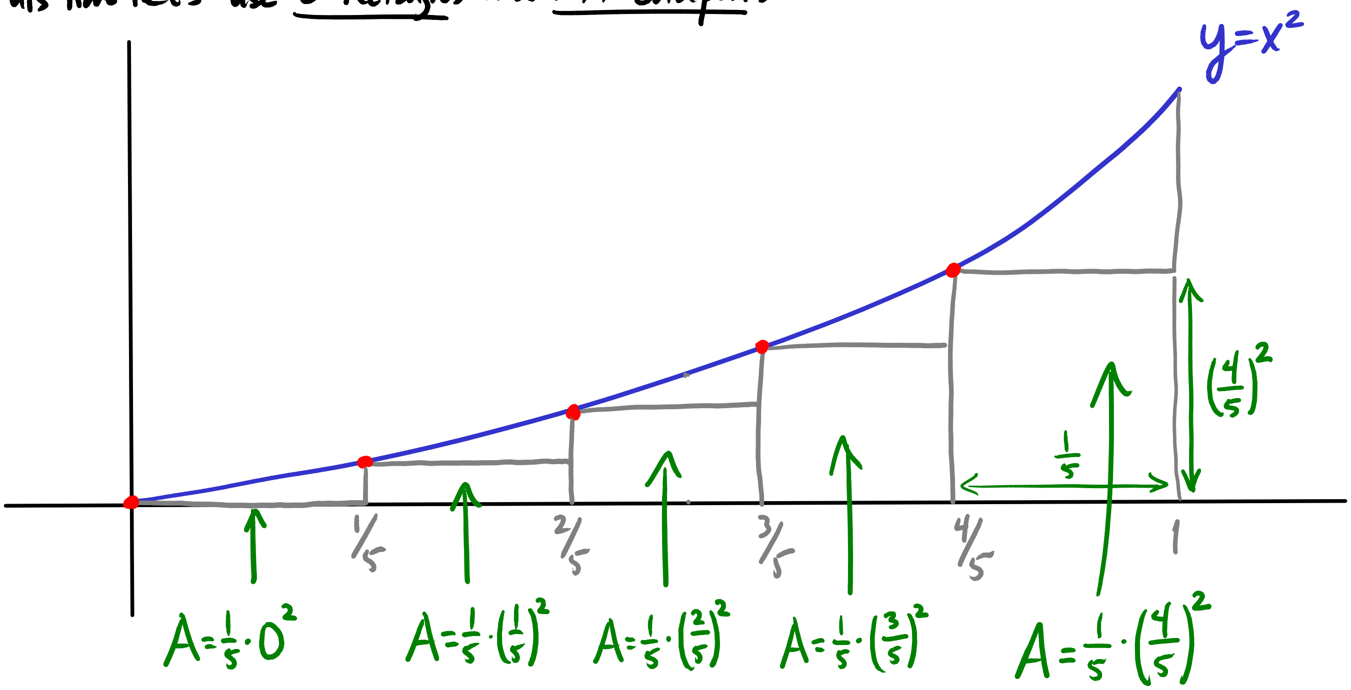
## More on areas (Ch 5.1)

Interested in the area under the curve $y=x^2$, between $x=0$ and $x=1$.




Answer depends exactly how you estimate.
Last time we used 4 rectangles and right endpoints as sample points. Got estimate $\frac{15}{32}$.
Write that answer $R_4=\frac{15}{32}$ (R="right", 4="4 rectangles").

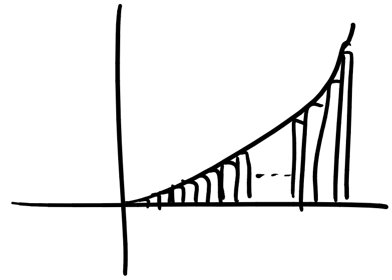
This time let's use 5 rectangles and left endpoints:



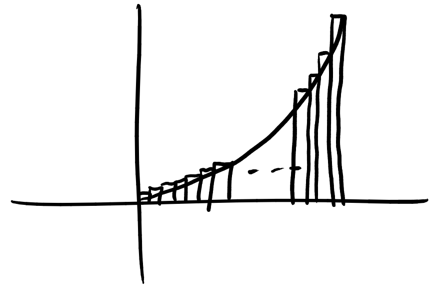

Total area $L_5=\frac{1}{5}\cdot\left(0^2+\left(\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2+\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2\right)$

Now suppose we used 100 rectangles. We would get

$$L_{100} = \frac{1}{100} \cdot \left(0^2 + \left(\frac{1}{100}\right)^2 + \left(\frac{2}{100}\right)^2 + \cdots + \left(\frac{99}{100}\right)^2\right) = 0.3283500$$



$$R_{100} = \frac{1}{100} \cdot \left(\left(\frac{1}{100}\right)^2 + \left(\frac{2}{100}\right)^2 + \cdots + (1)^2\right) = 0.3383500$$



| $n$ | $L_n$ | $R_n$ |
| --- | --- | --- |
| 10 | 0.2850000 | 0.3850000 |
| 100 | 0.3283500 | 0.3385000 |
| 1000 | 0.3328335 | 0.3338335 |

Indeed, as $n \to \infty$, both $L_n$ and $R_n$ approach $\frac{1}{3}$: $\left[\text{e.g. } R_n = \frac{(n+1)(2n+1)}{6n^2}\right]$

$$\lim_{n\to\infty} L_n = \frac{1}{3} \qquad \lim_{n\to\infty} R_n = \frac{1}{3}$$

$\frac{1}{3}$ is the exact area under the graph $y = x^2$ between $x=0$ and $x=1$.

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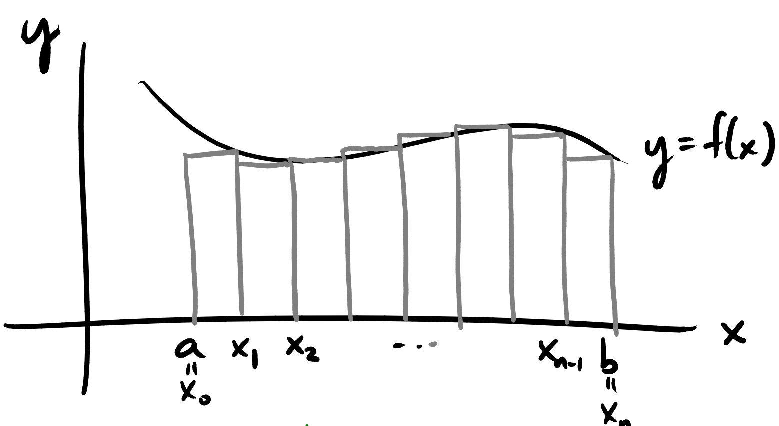
For a general function $f(x)$, we can calculate the area similarly:




Width $\Delta x = \frac{b-a}{n}$

Heights: $f(x_1), f(x_2), \ldots, f(x_n)$ [right endpoints] where $x_i = a + i\Delta x$

$\Rightarrow$ area estimate $R_n = \Delta x \left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]$

Another convenient notation: ("sigma notation")

The symbol $\sum_{i=1}^{n} f(x_i)$ means $f(x_1)+\cdots+f(x_n)$.

**Example:** Calculate $\sum_{i=1}^{4} i^2$.

$$\sum_{i=1}^{4} i^2 = 1^2+2^2+3^2+4^2 = \underline{\underline{30}}.$$

**Example:** Write $\frac{2^3}{n}+\frac{4^3}{n}+\frac{6^3}{n}+\cdots+\frac{(2n)^3}{n}$ in sigma notation.

$$\frac{2^3}{n}+\frac{4^3}{n}+\frac{6^3}{n}+\cdots+\frac{(2n)^3}{n} = \underline{\underline{\sum_{i=1}^{n}\frac{(2i)^3}{n}}}$$

In this notation, $R_n = \Delta x \sum_{i=1}^{n} f(x_i)$

And $L_n = \Delta x \sum_{i=1}^{n} f(x_{i-1})$

The actual area is $A = \lim_{n\to\infty} R_n$

or $A = \lim_{n\to\infty} L_n$ (both are the same!)

**Example:** Let $A$ be the area of the region under the graph of $f(x)=\sin^2 x$ between $x=\frac{\pi}{4}$ and $x=\frac{3\pi}{4}$. Using right endpoints as sample points,

- Write a formula for $A$ as a limit.

$a=\frac{\pi}{4}$, $b=\frac{3\pi}{4}$

$$\Delta x = \frac{b-a}{n} = \frac{\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)}{n} = \frac{\pi}{2n}$$

$$x_i = a + i\Delta x = \frac{\pi}{4}+i\frac{\pi}{2n}$$

$$R_n = \Delta x \sum_{i=1}^{n} f(x_i) = \frac{\pi}{2n}\sum_{i=1}^{n}\sin^2\left(\frac{\pi}{4}+i\frac{\pi}{2n}\right)$$

$$A = \lim_{n\to\infty} R_n = \underline{\underline{\lim_{n\to\infty}\left[\frac{\pi}{2n}\sum_{i=1}^{n}\sin^2\left(\frac{\pi}{4}+i\frac{\pi}{2n}\right)\right]}}$$

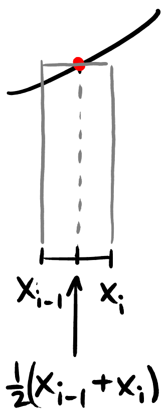
- Estimate $A$ using 3 rectangles.

$$R_3 = \frac{\pi}{6} \sum_{i=1}^{3} \sin^2\left(\frac{\pi}{4} + i\frac{\pi}{6}\right)$$

$$= \frac{\pi}{6}\left[\sin^2\left(\frac{\pi}{4}+\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{4}+\frac{2\pi}{6}\right) + \sin^2\left(\frac{\pi}{4}+\frac{3\pi}{6}\right)\right]$$

$$\approx 1.23885$$

You can also estimate area using other sample points, e.g. the midpoints of the intervals.




The limit $n \to \infty$ of the estimated area will still be the exact area.

(As long as $f$ is continuous!)