Lecture 10

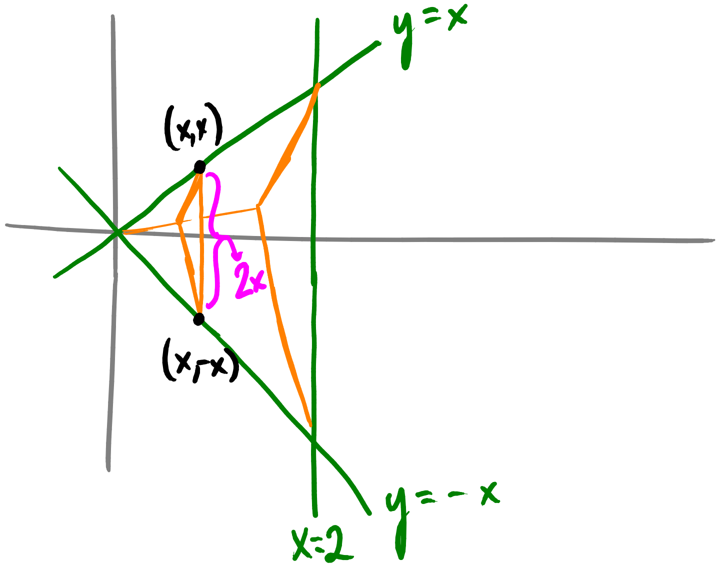
10 Feb 2010

Last time: volumes

$$V = \int_a^b dx \, A(x)$$

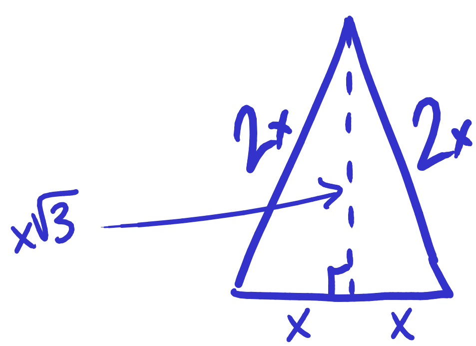
area of cross section of the solid at fixed x

Ex Calculate the volume of a solid whose base is the region between $y=x$, $y=-x$, and $x=2$, and whose cross sections at fixed x are equilateral triangles.



$$V = \int_0^2 A(x)\,dx$$

$A(x)$ = area of equilateral Δ with side length $2x$



area $= \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(2x)(x\sqrt{3})$

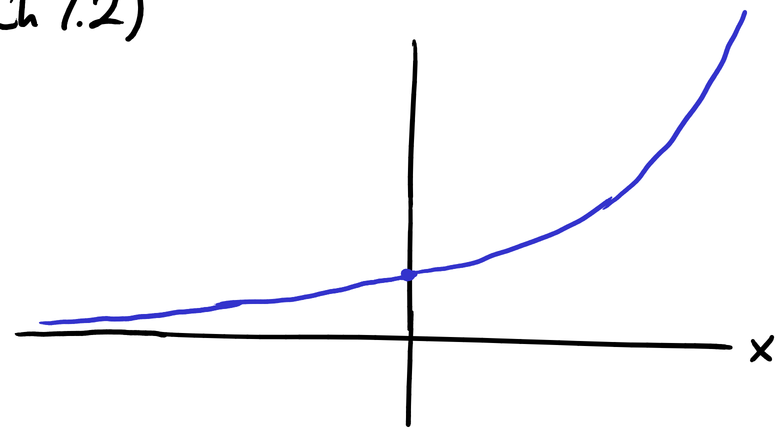
$= x^2\sqrt{3}$

so $V = \int_0^2 x^2\sqrt{3}\,dx = \frac{8}{3}\sqrt{3}$

Exponential functions (Ch 7.2)

$f(x) = e^x$

$\frac{d}{dx} e^x = e^x$



So also $\int e^x \, dx = e^x + C$.

Ex $\int x^2 e^{x^3} \, dx = ?$

Put $u = x^3$.
Then $du = 3x^2 \, dx$
$dx = \frac{du}{3x^2}$

So $\int x^2 e^{x^3} \, dx = \int x^2 e^u \frac{du}{3x^2} = \frac{1}{3} \int e^u \, du = \frac{1}{3}(e^u + C)$
$= \frac{1}{3}(e^{x^3} + C)$.

Ex $\int e^x \sqrt{1+e^x} \, dx = ?$

Put $u = 1 + e^x$
then $du = e^x \, dx$
so $dx = \frac{du}{e^x}$

so $\int e^x \sqrt{1+e^x} \, dx = \int e^x \sqrt{u} \frac{du}{e^x} = \int \sqrt{u} \, du = \frac{2}{3} u^{3/2} + C$
$= \frac{2}{3}(1+e^x)^{3/2} + C$

Also remember how to simplify exponents:

$e^a e^b = e^{a+b}$ e.g. $e^{3x^2} e^{-4x} = e^{3x^2 - 4x}$ ..

$(e^a)^b = e^{ab}$ e.g. $(e^{2x+1})^3 = e^{6x+3}$

Ex $\displaystyle\int \frac{(1+e^x)^2}{e^{\frac{1}{2}x}}\,dx = ?$

Multiply out:

$$\int \frac{(1+e^x)^2}{e^{\frac{1}{2}x}}\,dx = \int \frac{1+2e^x+(e^x)^2}{e^{\frac{1}{2}x}}\,dx = \int \frac{1+2e^x+e^{2x}}{e^{\frac{1}{2}x}}\,dx$$

$$= \int e^{-\frac{1}{2}x} + 2e^{\frac{1}{2}x} + e^{\frac{3}{2}x}\,dx \qquad \frac{1}{e^{\frac{1}{2}x}} = e^{-\frac{1}{2}x}$$

Could do by u-substitution on each term separately...

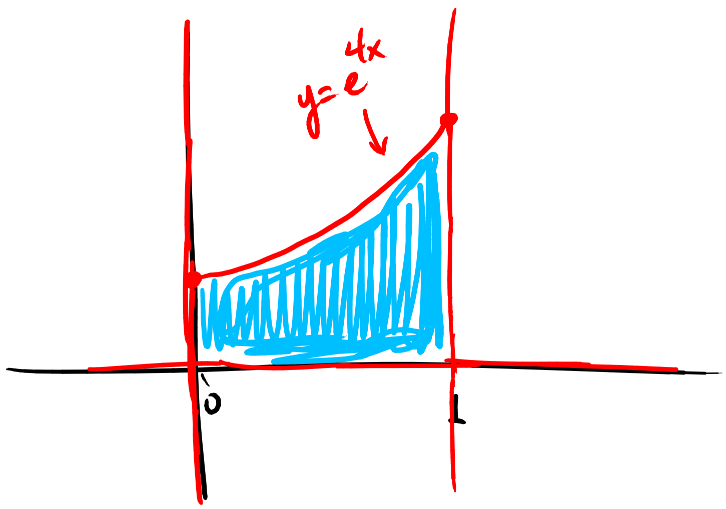
Or, use a shortcut $\displaystyle\int e^{bx}\,dx = \frac{1}{b}e^{bx} + C$ (when b is a constant)

to get

$$= (-2)e^{-\frac{1}{2}x} + (2)2e^{\frac{1}{2}x} + \frac{2}{3}e^{\frac{3}{2}x} + C$$

$$= \underline{\underline{-2e^{-\frac{1}{2}x} + 4e^{\frac{1}{2}x} + \frac{2}{3}e^{\frac{3}{2}x} + C}}$$

Ex Find the volume of the solid obtained by rotating the region bounded by $\left[\begin{array}{l} y=0 \\ x=0 \\ y=e^{4x} \\ x=1 \end{array}\right]$ around the x-axis.



$$V = \int_0^1 dx\, A(x) \qquad A(x) = \pi\left(e^{4x}\right)^2 = \pi e^{8x}$$

$$= \int_0^1 dx\, \pi e^{8x}$$

$$= \frac{\pi}{8} e^{8x}\Big|_0^1 = \underline{\underline{\frac{\pi}{8}\left(e^8 - 1\right)}}$$

A caution about exponents:

We have 2 different rules

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} e^x = e^x \qquad \left[\underline{\text{not}}\ x e^{x-1}!\right]$$

What about something like $\frac{d}{dx}(x^x)$?

Remember $x = e^{\ln x}$. So $x^x = (e^{\ln x})^x = e^{x \ln x}$

$$\frac{d}{dx}(e^{x \ln x}) = e^{x \ln x}\left[\frac{d}{dx} x \ln x\right]$$

$$= e^{x \ln x}\left[\ln x + \frac{x}{x}\right]$$

$$= e^{x \ln x}\left[\ln x + 1\right]$$

$$= x^x\left[\ln x + 1\right]$$

Logarithms: (Ch 7.4)

$$e^{\ln x} = x$$

Rules for simplifying:

$$\ln a + \ln b = \ln(ab)$$
$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$
$$k \ln a = \ln a^k$$

Differentiating:

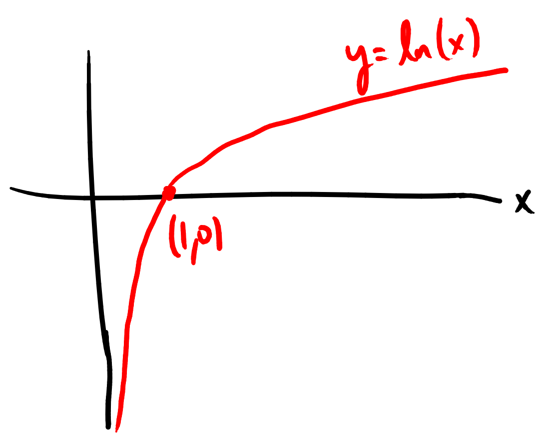
$$\frac{d}{dx}\ln x = \frac{1}{x}$$

NB: $\ln x$ only makes sense when $x > 0$!

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

[Or, can just write $\ln x$ if you know that $x > 0$.]

Ex
$$\int_{-4}^{-2} \frac{1}{x}\,dx = \ln|x|\Big|_{-4}^{-2} = \ln|-2| - \ln|-4|$$
$$= \ln 2 - \ln 4$$
$$= \ln \frac{2}{4}$$
$$= \underline{\underline{\ln \frac{1}{2}}}$$



$\ln(1) = 0$

Ex

$$\int \cot x \, dx = ?$$

$$= \int \frac{\cos x}{\sin x} \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$\left[\frac{du}{dx} = \cos x \right]$

$$= \int \frac{du}{u}$$

$$= \ln |u| + C$$

$$= \ln |\sin x| + C$$